Semidensities, Second-Class Constraints and Conversion in Anti-Poisson Geometry

K. Bering[1]
Institute for Theoretical Physics & Astrophysics
Masaryk University
Kotlářská 2
CZ-611 37 Brno
Czech Republic

April 25, 2008

Abstract

We consider Khudaverdian's geometric version of a Batalin-Vilkovisky (BV) operator Δ_E in the case of a degenerate anti-Poisson manifold. The characteristic feature of such an operator (aside from being a Grassmann-odd, nilpotent, second-order differential operator) is that it sends semidensities to semidensities. We find a local formula for the Δ_E operator in arbitrary coordinates. As an important application of this setup, we consider the Dirac antibracket on an antisymplectic manifold with antisymplectic second-class constraints. We show that the entire Dirac construction, including the corresponding Dirac BV operator Δ_{E_D}, exactly follows from conversion of the antisymplectic second-class constraints into first-class constraints on an extended manifold.

MCS number(s): 53A55; 58A50; 58C50; 81T70.
Keywords: Batalin-Vilkovisky Field-Antifield Formalism; Odd Laplacian; Anti-Poisson Geometry; Semidensity; Second-Class Constraints; Conversion.

[1]E-mail: bering@physics.muni.cz

Contents

1 Introduction

Consider an antisymplectic manifold $(M;E)$ with coordinates Γ^A. Such structure was first used by Batalin and Vilkovisky to quantize Lagrangian gauge theories [1, 2, 3]. In general, antisymplectic

geometry has many of the characteristic features of ordinary symplectic geometry, *e.g.* the Jacobi identity and the Darboux Theorem, but there are also important differences: There are no canonical volume form and no Liouville Theorem in antisymplectic geometry [4]. In the covariant Batalin-Vilkovisky (BV) formalism [5, 6] from around 1992 one is (among other things) instructed to make separate choices of a measure density $\rho=\rho(\Gamma)$ and a quantum action $W_{\rho}=W_{\rho}(\Gamma)$. However, the division into measure and action part is to a large extent an arbitrary division, *i.e.* it is always possible to shift parts of the measure ρ into the action W_{ρ} and vice versa. It is only a particular combination of these two quantities, namely the Boltzmann semidensity

$$\exp[\frac{i}{\hbar}W_{E}] \;\equiv\; \sqrt{\rho}\exp[\frac{i}{\hbar}W_{\rho}] \tag{1.1}$$

that enters the physical partition function $\mathcal{Z}$. For instance, if there exist global Darboux coordinates $\Gamma^{A}=\{\phi^{\alpha};\phi^{*}_{\alpha}\}$, the partition function reads

$$\mathcal{Z} \;=\; \int[d\phi]\,\exp[\frac{i}{\hbar}W_{E}]\Big|_{\phi^{*}=\frac{\partial\psi}{\partial\phi}} \;, \tag{1.2}$$

where $\psi=\psi(\phi)$ is the gauge fermion. (More generally, the partition function $\mathcal{Z}$ is described by the so-called W-X formalism [7, 8].) The field-antifield formalism was reformulated in Ref. [9] entirely in the minimal language of semidensities, which skips ρ altogether. According to this minimal approach, the Boltzmann semidensity $\exp[\frac{i}{\hbar}W_{E}]$ should satisfy the Quantum Master Equation

$$\Delta_{E}\exp[\frac{i}{\hbar}W_{E}] \;=\; 0 \tag{1.3}$$

to ensure independence of gauge-fixing. Here Δ_{E} is Khudaverdian's BV operator, which takes semidensities to semidensities, cf. Ref. [8, 10, 11, 12, 13] and Definition 2.3 below. Of course, the density ρ may always be re-introduced to compare with the 1992 formulation. In doing so, for an arbitrary choice of ρ,

1. the Boltzmann semidensity $\exp[\frac{i}{\hbar}W_{E}]$ descend to a Boltzmann scalar $\exp[\frac{i}{\hbar}W_{\rho}]=\exp[\frac{i}{\hbar}W_{E}]/\sqrt{\rho}$,
2. the Δ_{E} operator descend to a (not necessarily nilpotent) odd Laplacian Δ_{ρ}, which takes scalars to scalars, cf. Definition 2.2 below; and
3. the Quantum Master Eq. (1.3) descend to the Modified Quantum Master Equation

$$(\Delta_{\rho}+\nu_{\rho})\exp[\frac{i}{\hbar}W_{\rho}] \;=\; 0\;, \tag{1.4}$$

   where ν_{ρ} is an odd scalar, cf. Definition 2.8 below.

We emphasize that this construction works for any ρ. However, to arrive at the 1992 formulation [5, 6], which has $\nu_{\rho}=0$ and a nilpotent odd Laplacian $\Delta_{\rho}^{2}=0$, one should impose conditions on ρ.

The paper is organized as follows. Anti-Poisson geometry is reviewed in Section 2. The notions of compatible two-form fields and bi-Darboux coordinates are introduced in Subsection 2.1. A new Theorem 2.1 provides necessary and sufficient conditions for the existence of bi-Darboux coordinates. The definition of the Δ_{E} operator for a degenerate anti-Poisson structure E is given using both Darboux and general coordinates in Subsection 2.3 and 2.4, respectively. The Δ_{E} formula in general coordinates does require the existence of a compatible two-form fields, however, it does not matter which compatible two-form field that is used (in case there is more than one choice), cf. Lemma 2.7. All information about how the Δ_{E} operator acts on semidensities can be packed into a Grassmann-odd

scalar quantity ν_ρ, which already appeared in eq. (1.4) above. The odd scalar ν_ρ is important, because in practice it is easier to handle a scalar object rather than the full second-order differential operator Δ_E, and hence many of the ensuring arguments is performed using ν_ρ. The Dirac antibracket is an important application of the geometric setup from Section 2, since it always admits a compatible two-form field. Antisymplectic second-class constraints and the Dirac antibracket [6, 8, 14] are reviewed in Subsection 3.1. A Proposition 3.1 in Subsection 3.2 provides a useful formula for the corresponding Dirac odd scalar ν_{ρ,E_D}. Subsection 3.4 discusses the stability of the Dirac construction under reparameterizations of the second-class constraints. In Section 4 the Dirac construction is derived via conversion [15, 16, 17, 18, 19] of the antisymplectic second-class constraints into first-class constraints on an extended manifold. As an application of the construction to Batalin-Vilkovisky quantization, the corresponding Dirac and extended partition functions are provided in Subsections 3.6 and 4.7, respectively. Finally, Section 5 contains our conclusions.

General remark about notation. We have two types of grading: A Grassmann grading ε and an exterior form degree p. The sign conventions are such that two exterior forms ξ and η, of Grassmann parity ε_ξ, ε_η and exterior form degree p_ξ, p_η, respectively, commute in the following graded sense

$$\eta \wedge \xi \;=\; (-1)^{\varepsilon_\xi\varepsilon_\eta + p_\xi p_\eta}\,\xi \wedge \eta \tag{1.5}$$

inside the exterior algebra. We will often not write the exterior wedges "$\wedge$" explicitly.

2 Anti-Poisson Geometry

2.1 Antibracket and Compatible Two-Form

We consider an anti-Poisson manifold $(M;E^{AB})$ with a (possibly degenerate) antibracket

$$(F,G) \;=\; (F\overleftarrow{\partial}{}^r_A)E^{AB}(\overrightarrow{\partial}{}^l_B G) \;=\; -(-1)^{(\varepsilon_F+1)(\varepsilon_G+1)}(G,F)\;, \qquad \overrightarrow{\partial}{}^l_A \;\equiv\; \frac{\overrightarrow{\partial}{}^l}{\partial\Gamma^A}\;, \tag{2.1}$$

Here the Γ^A's denote local coordinates of Grassmann parity $\varepsilon_A \equiv \varepsilon(\Gamma^A)$, and $E^{AB}=E^{AB}(\Gamma)$ is the local matrix representation of the anti-Poisson structure E. The Jacobi identity

$$\sum_{\text{cycl. } F,G,H} (-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}(F,(G,H)) \;=\; 0 \tag{2.2}$$

reads in local coordinates

$$\sum_{\text{cycl. } A,B,C} (-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}E^{AD}(\overrightarrow{\partial}{}^l_D E^{BC}) \;=\; 0\;. \tag{2.3}$$

The main new feature (as compared to Ref. [9]) is that the anti-Poisson structure E^{AB} could be degenerate. There is an anti-Poisson analogue of Darboux's Theorem that states that locally, if the rank of E^{AB} is constant, there exist Darboux coordinates $\Gamma^A=\{\phi^\alpha;\phi^*_\alpha;\Theta^a\}$, such that the only non-vanishing antibrackets between the coordinates are $(\phi^\alpha,\phi^*_\beta)=\delta^\alpha_\beta=-(\phi^*_\beta,\phi^\alpha)$. In other words, the Jacobi identity is the integrability condition for the Darboux coordinates. The variables ϕ^α, ϕ^*_α and Θ^a are called *fields*, *antifields* and *Casimirs*, respectively.

We shall assume that the anti-Poisson manifold $(M;E^{AB})$ admits a globally defined odd two-form field E_{AB} with lower indices that is *compatible* with the anti-Poisson structure E^{AB} in the sense that

$$E^{AB}E_{BC}E^{CD} \;=\; E^{AD}\;,$$

$$E_{AB}E^{BC}E_{CD} \;=\; E_{AD}\ . \tag{2.4}$$

As always, the matrices E^{AB} and E_{AB} are assumed to have the Grassmann gradings

$$\varepsilon(E^{AB}) \;=\; \varepsilon_A + \varepsilon_B + 1 \;=\; \varepsilon(E_{AB})\ , \tag{2.5}$$

and the skew-symmetries

$$\begin{aligned} E^{BA} &= -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}E^{AB}\ , \\ E_{BA} &= -(-1)^{\varepsilon_A\varepsilon_B}E_{AB}\ . \end{aligned} \tag{2.6}$$

The odd two-form field can be written as

$$E \;=\; \frac{1}{2}d\Gamma^A\ E_{AB}\ d\Gamma^B \;=\; -\frac{1}{2}E_{AB}\ d\Gamma^B\ d\Gamma^A\ . \tag{2.7}$$

The two-form field E_{AB} would be closed if

$$dE \;=\; 0\ , \tag{2.8}$$

or equivalently, with all the indices written out, if

$$\sum_{\mathrm{cycl.}\ A,B,C}(-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_A}E_{BC}) \;=\; 0\ . \tag{2.9}$$

A closed degenerate two-form is called a pre-antisymplectic structure. In the non-degenerate case, the matrix E_{AB} from eq. (2.4) would be a closed antisymplectic two-form field and the inverse of the anti-Poisson structure E^{AB}. In the degenerate case, there is in general *not a unique* matrix E_{AB} fulfilling eqs. (2.4), (2.5) and (2.6), and there is *no* reason for it to be closed. In Darboux coordinates $\Gamma^A=\{\phi^\alpha;\phi^*_\alpha;\Theta^a\}$, there is still a freedom in a compatible two-form

$$E \;=\; d\phi^*_\alpha\wedge d\phi^\alpha + d\Theta^a M_{a\alpha}\wedge d\phi^\alpha + d\phi^*_\alpha N^\alpha{}_a\wedge d\Theta^a + d\Theta^a M_{a\alpha}N^\alpha{}_b\wedge d\Theta^b \tag{2.10}$$

given by two arbitrary matrices $M_{a\alpha}=M_{a\alpha}(\Gamma)$ and $N^\alpha{}_a=N^\alpha{}_a(\Gamma)$. A Darboux coordinate system $\Gamma^A=\{\phi^\alpha;\phi^*_\alpha;\Theta^a\}$ is called a *bi-Darboux* coordinate system, if the two-form is just $E=d\phi^*_\alpha\wedge d\phi^\alpha$, *i.e.* if both the matrices $M_{a\alpha}=0$ and $N^\alpha{}_a=0$ in eq. (2.10) are equal to zero. In short, the Γ^A's are bi-Darboux coordinates, if both matrices E^{AB} and E_{AB} with upper and lower indices are on standard form.

Theorem 2.1 *Given an anti-Poisson manifold $(M;E^{AB})$ with a compatible two-form field E_{AB}. Then there locally exist bi-Darboux coordinates if and only if the two-form field E_{AB} is closed.*

There is a similar Bi-Darboux Theorem for even Poisson structures. A proof of Theorem 2.1 is given in Appendix A. One can define a projection as

$$P^A{}_C \;\equiv\; E^{AB}E_{BC}\ , \tag{2.11}$$

or equivalently,

$$P_A{}^C \;\equiv\; E_{AB}E^{BC} \;=\; (-1)^{\varepsilon_A(\varepsilon_C+1)}P^C{}_A\ . \tag{2.12}$$

It follows from property (2.4) that

$$P^A{}_B P^B{}_C \;=\; P^A{}_C\ . \tag{2.13}$$

In the non-degenerate case $P^A{}_B=\delta^A_B=P_B{}^A$.

2.2 Odd Laplacian Δ_ρ on Scalars

Recall that a scalar function $F = F(\Gamma)$, a density $\rho = \rho(\Gamma)$ and a semidensity $\sigma = \sigma(\Gamma)$ are by definition quantities that transform as

$$F \longrightarrow F' = F , \qquad \rho \longrightarrow \rho' = \frac{\rho}{J} , \qquad \sigma \longrightarrow \sigma' = \frac{\sigma}{\sqrt{J}} , \tag{2.14}$$

respectively, under general coordinate transformations $\Gamma^A \to \Gamma'^A$, where $J \equiv \mathrm{sdet}\frac{\partial \Gamma'^A}{\partial \Gamma^B}$ denotes the Jacobian. We shall ignore the global issues of orientation and choice of square root. Also we assume that densities ρ are invertible.

Definition 2.2 *Given a choice of a density* ρ*, the* **odd Laplacian** Δ_ρ *is defined as [6]*

$$\Delta_\rho \equiv \frac{(-1)^{\varepsilon_A}}{2\rho} \overrightarrow{\partial^l_A} \rho E^{AB} \overrightarrow{\partial^l_B} . \tag{2.15}$$

This Grassmann-odd, second-order operator takes scalar functions to scalar functions. In situations with more than one anti-Poisson structure E^{AB}, we shall sometimes use the slightly longer notation $\Delta_\rho \equiv \Delta_{\rho,E}$ to acknowledge that it depends on two inputs: ρ and E^{AB}. The odd Laplacian Δ_ρ "differentiates" the antibracket $(\cdot,\cdot)$, *i.e.* the following Leibniz-type rule holds

$$\Delta_\rho(F,G) = (\Delta_\rho F, G) + (-1)^{(\varepsilon_F+1)}(F, \Delta_\rho G) . \tag{2.16}$$

For further information on this important operator, see Ref. [8, 9] and Subsection 2.5 below.

2.3 The Δ_E Operator on Semidensities

There is an another important Grassmann-odd, nilpotent, second-order operator Δ_E that depends only on the anti-Poisson structure E^{AB}. Contrary to the odd Laplacian $\Delta_\rho \equiv \Delta_{\rho,E}$ of last Subsection 2.2, the Δ_E operator does *not* rely on a choice of density ρ. The caveat is that while the odd Laplacian Δ_ρ takes scalars to scalars, the Δ_E operator takes semidensities to semidensities of opposite Grassmann parity. Equivalently, the Δ_E operator transforms as

$$\Delta_E \longrightarrow \Delta'_E = \frac{1}{\sqrt{J}} \Delta_E \sqrt{J} \tag{2.17}$$

under general coordinate transformations $\Gamma^A \to \Gamma'^A$, cf. eq. (2.14). It is defined as follows:

Definition 2.3 *Let there be given an anti-Poisson manifold* $(M;E)$*. In Darboux coordinates* Γ^A*, the* Δ_E **operator** *is defined on a semidensity* σ *as [8, 10, 11, 12, 13]*

$$(\Delta_E \sigma) \equiv (\Delta_1 \sigma) , \tag{2.18}$$

where Δ_1 *denotes the expression (2.15) for the odd Laplacian* $\Delta_{\rho=1}$ *with* ρ *replaced by* 1.

It is implicitly understood in eq. (2.18) that the formula for the Δ_1 operator (2.15) and the semidensity σ both refer to the same Darboux coordinates Γ^A. The parentheses in eq. (2.18) indicate that the equation should be understood as an equality among semidensities (in the sense of zeroth-order differential operators) rather than an identity among differential operators. The Definition 2.3 does not depend on the Darboux coordinate system being used, due to the following Lemma 2.4:

Lemma 2.4 *When using the Definition 2.3, the* $(\Delta_E\sigma)$ *transforms as a semidensity under (anti-canonical) transformations between sets of Darboux coordinates.*

Thus the Δ_E operator is a well-defined operator on an open cover of Darboux neighborhoods. Within this cover, the Δ_E is indirectly defined in non-Darboux coordinates by use of the transformation property (2.17). Lemma 2.4 was first proven in the non-degenerate case in Ref. [13] and in the degenerate case in Ref. [8]. We shall also give an independent proof in the next Subsection 2.4, cf. Lemma 2.6 below. In some cases the Δ_E operator may be extended to singular points (*i.e.* points where the rank of the anti-Poisson tensor E^{AB} jumps) by continuity.

Working in Darboux coordinates, it is obvious that the Δ_E operator super-commutes with itself, because the Γ^A-derivatives have no Γ^A's to act on when E^{AB} is on Darboux form. Therefore Δ_E is nilpotent,

$$\Delta_E^2 \;=\; \frac{1}{2}[\Delta_E, \Delta_E] \;=\; 0 \; . \tag{2.19}$$

Same sort of reasoning shows that $\Delta_E = \Delta_E^T$ is symmetric.

2.4 The Δ_E Operator in General Coordinates

We now give a definition of the Δ_E operator that does not refer to Darboux coordinates.

Definition 2.5 *Given an anti-Poisson manifold* $(M; E^{AB})$ *that admits a compatible two-form field* E_{AB}*. In arbitrary coordinates* Γ^A*, the* Δ_E *operator is defined as*

$$(\Delta_E\sigma) \;\equiv\; (\Delta_1\sigma) + \left(\frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12}\right)\sigma \; , \tag{2.20}$$

where

$$\nu^{(1)} \;\equiv\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}E^{AB}) \; , \tag{2.21}$$

$$\nu^{(2)} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}E^{CD}) \; , \tag{2.22}$$

$$\nu^{(3)} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{BA}) \; , \tag{2.23}$$

$$\nu^{(4)} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A \; , \tag{2.24}$$

$$\begin{aligned}\nu^{(5)} \;&\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}E^{CF})P_F{}^D \\ &=\; (-1)^{(\varepsilon_A+1)\varepsilon_B}E^{AD}(\overrightarrow{\partial^l_D}E^{BC})(\overrightarrow{\partial^l_C}E_{AF})P^F{}_B \; .\end{aligned} \tag{2.25}$$

Notice that in Darboux coordinates, where E^{AB} is constant, *i.e.* independent of the coordinates Γ^A, the last five terms $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ become zero. Hence the new Definition 2.5 agrees in Darboux coordinates with the previous Definition 2.3. The benefit of the new Definition 2.5 is that one now have an explicit formula for Δ_E in an arbitrary coordinate system. The full justification of Definition 2.5 is provided by the following Lemma 2.6 and Lemma 2.7.

Lemma 2.6 *When using the new Definition 2.5, the* $(\Delta_E\sigma)$ *transforms as a semidensity under general coordinate transformations.*

Lemma 2.7 *When using the new Definition 2.5, the* $(\Delta_E\sigma)$ *does not depend on the compatible two-form field* E_{AB} *used.*

The explicit formula (2.20) and Lemma 2.6 are the main results of Section 2.

Proof of Lemma 2.7: The two-form field E_{AB} enters only the Definition 2.5 via $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$. Assuming the Lemma 2.6, *i.e.* that the behavior (2.17) under general coordinate transformations has already been established, one may, in particular, go to Darboux coordinates, where $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ vanish identically.

$\square$

To prove Lemma 2.6 we shall first reformulate it as an equivalent Lemma 2.9, cf. below. We shall also only explicitly consider the case where σ is invertible to simplify the presentation. (The non-invertible case is fundamentally no different.) In the invertible case, we customarily write the semidensity $\sigma = \sqrt{\rho}$ as a square root of a density ρ, and define a Grassmann-odd quantity ν_ρ as follows.

Definition 2.8 *The* **odd scalar** ν_ρ *is defined as*

$$\nu_\rho \;\equiv\; \frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) \;=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} \;, \qquad (2.26)$$

where $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$, $\nu^{(5)}$ *are given in eqs. (2.21)–(2.25), and the quantity* $\nu_\rho^{(0)}$ *is given as*

$$\nu_\rho^{(0)} \;\equiv\; \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) \;. \qquad (2.27)$$

In situations with more than one anti-Poisson structure E^{AB}, we shall sometimes use the slightly longer notation $\nu_\rho \equiv \nu_{\rho,E}$. By dividing both sides of the definition (2.20) with the semidensity σ, one may reformulate the content of Lemma 2.6 as:

Lemma 2.9 *The Grassmann-odd quantity* ν_ρ *is a scalar, i.e. it does not depend on the coordinate system.*

We shall give two independent proofs of this important Lemma 2.9; one relying on Darboux Theorem and the other using infinitesimal coordinate transformations.

Proof of Lemma 2.9 using a Darboux coordinate patch: It is enough to consider how ν_ρ behaves on coordinate transformations $\Gamma_0^A \to \Gamma^A$ between Darboux coordinates Γ_0^A and general coordinates Γ^A. (An arbitrary coordinate transformation between two general coordinate patches can always be split into two successive coordinate transformations of the above kind by inserting a third Darboux coordinate patch in between.) The idea is now to first consider the expression (2.26) for ν_ρ in the Γ^A coordinate system, and decompose it in building blocks that refer to the Darboux coordinates Γ_0^A, *e.g.*

$$E^{AD} \;=\; (\Gamma^A \frac{\overleftarrow{\partial^r}}{\partial\Gamma_0^B}) E_0^{BC} (\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^C}\Gamma^D) \;, \qquad E_{AD} \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\Gamma^a}\Gamma_0^B) E^0_{BC} (\Gamma_0^C \frac{\overleftarrow{\partial^r}}{\partial\Gamma^D}) \;, \qquad \rho \;=\; \frac{\rho_0}{J} \;. \qquad (2.28)$$

Here $J \equiv \mathrm{sdet}(\partial\Gamma^A/\partial\Gamma_0^B)$ denotes the Jacobian of the coordinate transformations $\Gamma_0^A \to \Gamma^A$. Recall that the two-form field E^0_{BC} is not necessarily constant in the Darboux coordinates Γ_0^A, cf eq. (2.10). By straightforward calculation, one gets

$$\nu_\rho^{(0)} \equiv \frac{1}{\sqrt{\rho}}(\Delta_{1,E}\sqrt{\rho}) = \frac{1}{\sqrt{\rho}}(\Delta_{J,E_0}\sqrt{\rho}) = \frac{1}{\sqrt{\rho_0}}(\Delta_{1,E_0}\sqrt{\rho_0}) - \frac{1}{\sqrt{J}}(\Delta_{1,E_0}\sqrt{J}) , \tag{2.29}$$

$$\nu^{(1)} = \frac{8}{\sqrt{J}}(\Delta_{1,E_0}\sqrt{J}) - (-1)^{\varepsilon_B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^A}\Gamma^B,\ \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma_0^A) , \tag{2.30}$$

$$\nu^{(2)} = -(-1)^{\varepsilon_B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^A}\Gamma^B,\ \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma_0^A) - 2(-1)^{\varepsilon_B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^A}\Gamma^B,\ \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma_0^C)P_C^{0,A} , \tag{2.31}$$

$$\begin{aligned}\nu^{(3)} = {}& 3(-1)^{\varepsilon_B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^A}\Gamma^B,\ \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma_0^C)P_C^{0,A} \\ & -(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}(\Gamma_0^A\frac{\overleftarrow{\partial^r}}{\partial\Gamma^B})(\Gamma^B\frac{\overleftarrow{\partial^r}}{\partial\Gamma_0^C},\ E^0_{AD})E_0^{DC} ,\end{aligned} \tag{2.32}$$

$$\begin{aligned}\nu^{(4)} = {}& (-1)^{\varepsilon_B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^A}\Gamma^B,\ \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma_0^C)P_C^{0,A} + 2(-1)^{\varepsilon_A\varepsilon_C}P_A^{0,B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^B}\Gamma^C,\ \Gamma_0^A\frac{\overleftarrow{\partial^r}}{\partial\Gamma^D})P^D{}_C \\ & -(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}(\Gamma_0^A\frac{\overleftarrow{\partial^r}}{\partial\Gamma^B})(\Gamma^B\frac{\overleftarrow{\partial^r}}{\partial\Gamma_0^C},\ E^0_{AD})E_0^{DC} ,\end{aligned} \tag{2.33}$$

$$\nu^{(5)} = -2(-1)^{\varepsilon_B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^A}\Gamma^B,\ \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma_0^C)P_C^{0,A} - (-1)^{\varepsilon_A\varepsilon_C}P_A^{0,B}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^B}\Gamma^C,\ \Gamma_0^A\frac{\overleftarrow{\partial^r}}{\partial\Gamma^D})P^D{}_C . \tag{2.34}$$

The last equality in eq. (2.29) is a non-trivial property of the odd Laplacian. It is now easy to check that all but one of the above terms on the right-hand sides of eqs. (2.29)–(2.34) cancel in the pertinent linear combination (2.26), *i.e.*

$$\nu_\rho = \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} = \frac{1}{\sqrt{\rho_0}}(\Delta_{1,E_0}\sqrt{\rho_0}) . \tag{2.35}$$

The surviving term, on the other hand, is just the definition for ν_ρ in the Darboux coordinates Γ_0^A.

□

Proof of Lemma 2.9 using infinitesimal coordinate transformations: Under an arbitrary infinitesimal coordinate transformation $\delta\Gamma^A = X^A$, one calculates

$$\delta\nu_\rho^{(0)} = -\frac{1}{2}\Delta_1\mathrm{div}_1 X , \tag{2.36}$$

$$\delta\nu^{(1)} = 4\Delta_1\mathrm{div}_1 X + (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_C}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C) , \tag{2.37}$$

$$\delta\nu^{(2)} = (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_D}E^{AB})\left(2P_B{}^C(\overrightarrow{\partial^l_C}\overrightarrow{\partial^l_A}X^D) + (\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C)P_C{}^D\right) , \tag{2.38}$$

$$\begin{aligned}\delta\nu^{(3)} = {}& (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\left((\overrightarrow{\partial^l_D}X^B\overleftarrow{\partial^r_F})E^{FA} - (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}(\overrightarrow{\partial^l_D}X^A\overleftarrow{\partial^r_F})E^{FB}\right) \\ & -\frac{3}{2}(-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C) ,\end{aligned} \tag{2.39}$$

$$\delta\nu^{(4)} = -2(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B}X^C)P_C{}^D(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A$$

$$
\begin{aligned}
& +(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^B\overleftarrow{\partial^r_F})E^{FA} \\
& +(-1)^{(\varepsilon_B+1)\varepsilon_F}P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^F\overleftarrow{\partial^r_G})E^{GB} \\
& +\frac{1}{2}(-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C) \ , \qquad (2.40)
\end{aligned}
$$

$$
\begin{aligned}
\delta\nu^{(5)} \ = \ & -(-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^A\overleftarrow{\partial^r_F})E^{FB} \\
& +2(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B}X^C)P_C{}^D(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A \ . \qquad (2.41)
\end{aligned}
$$

A proof of eqs. (2.36) and (2.37) can be found in Ref. [9], and eqs. (2.38)–(2.41) are proven in Appendix B. One may verify that while the six constituents $\nu_\rho^{(0)}$, $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ separately have non-trivial transformation properties, the linear combination ν_ρ in eq. (2.26) is indeed a scalar.

□

The new Definition 2.5 is clearly symmetric $\Delta_E = \Delta_E^T$. To check explicitly in general coordinates that Δ_E is nilpotent is a straightforward (but admittedly tedious) exercise. However, since we have just proven that Δ_E behaves covariantly under general coordinate transformations, our previous proof of nilpotency from last Subsection 2.3 using Darboux coordinates suffices. To summarize:

Theorem 2.10 *The Δ_E operator (2.20) is nilpotent (2.19) if and only if the antibracket (2.1) satisfies the Jacobi identity (2.3).*

In the rest of the paper we will always assume that the Jacobi identity (2.3) is satisfied, and hence that the Δ_E operator (2.20) is nilpotent.

2.5 Nilpotency Condition for the odd Laplacian Δ_ρ

At this point it is instructive to recall the nilpotency condition for the odd Laplacian Δ_ρ, although we shall not assume that it is satisfied. It follows from the Jacobi identity (2.3) alone, that Δ_ρ^2 is a linear derivation, *i.e.* a first-order differential operator. The interplay between the two second-order differential operators Δ_E and Δ_ρ is perhaps best summarized by the following operator identity:

$$
\Delta_\rho + \nu_\rho \ = \ \frac{1}{\sqrt{\rho}}\Delta_E\sqrt{\rho} \ , \qquad (2.42)
$$

cf. eq. (5.9) of Ref. [9]. In words: Apart from the ν_ρ term the odd Laplacian Δ_ρ is the Δ_E operator dressed with a $\sqrt{\rho}$ factor. From this operator identity (2.42) and the nilpotency (2.19) of the Δ_E operator, one derives the explicit form of the linear derivation:

$$
\Delta_\rho^2 \ = \ (\nu_\rho, \ \cdot \ ) \ . \qquad (2.43)
$$

Therefore the nilpotency condition for Δ_ρ reads [8, 11]

$$
\Delta_\rho^2 \ = \ 0 \qquad \Leftrightarrow \qquad \nu_\rho \text{ is a Casimir.} \qquad (2.44)
$$

Let us also mention for later that if one acts with the operator identity (2.42) on a scalar function $\sqrt{F}$, one gets

$$
\nu_{\rho F} \ = \ \nu_\rho + \frac{1}{\sqrt{F}}(\Delta_\rho\sqrt{F}) \ . \qquad (2.45)
$$

2.6 Alternative Expressions

It is convenient to introduce

$$\nu^{(23)} \equiv \nu^{(2)}+\nu^{(3)} = (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}P_B{}^C)(\overrightarrow{\partial^l_C}E^{BA}) , \tag{2.46}$$

$$\begin{aligned}\nu^{(35)} \equiv \nu^{(3)}+\nu^{(5)} &= (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}P_B{}^C)P_C{}^D(\overrightarrow{\partial^l_D}E^{BA}) \\ &= (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}P^B{}_C)E^{CD}(\overrightarrow{\partial^l_D}P_B{}^A) ,\end{aligned} \tag{2.47}$$

$$\nu^{(45)} \equiv \nu^{(4)}+\nu^{(5)} = (-1)^{\varepsilon_B}P_A{}^D(\overrightarrow{\partial^l_D}P_B{}^C)(\overrightarrow{\partial^l_C}E^{BA}) , \tag{2.48}$$

$$\nu^{(23)}_{(45)} \equiv \nu^{(23)}-\nu^{(45)} = (-1)^{\varepsilon_B(\varepsilon_D+1)}(\overrightarrow{\partial^l_A}P^B{}_C)E^{CD}(\overrightarrow{\partial^l_B}P_D{}^A) . \tag{2.49}$$

Then the Δ_E operator (2.20) may be re-written as

$$(\Delta_E\sigma) = (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{24}-\frac{\nu^{(23)}}{12}+\frac{\nu^{(35)}+\nu^{(45)}}{24}\right)\sigma , \tag{2.50}$$

$$= (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}+\nu^{(23)}-\nu^{(35)}+\nu^{(23)}_{(45)}}{24}\right)\sigma . \tag{2.51}$$

In the closed case (2.8) one may show that

$$\nu^{(35)}+\nu^{(45)} = 0 , \tag{2.52}$$

so that the Δ_E operator (2.50) simplifies to

$$(\Delta_E\sigma) = (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{8}-\frac{\nu^{(3)}}{12}\right)\sigma . \tag{2.53}$$

In the non-degenerate case, which is automatically closed, one also has

$$\nu^{(23)} = 0 , \tag{2.54}$$

so that the Δ_E operator (2.50) simplifies even further to

$$(\Delta_E\sigma) = (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{24}\right)\sigma , \tag{2.55}$$

in agreement with eq. (5.1) in Ref. [9].

3 Second-Class Constraints

3.1 Review of Dirac Antibracket

One of the most important examples of degenerate anti-Poisson structures is provided by the Dirac antibracket [6, 8, 14]. Consider a manifold $(M;E)$ with a non-degenerate anti-Poisson structure E^{AB} (called an antisymplectic phase space), and let a submanifold $\tilde{M}\equiv\{\Gamma\in M|\Theta(\Gamma)=0\}$ be the zero-locus of a set of constraints $\Theta^a=\Theta^a(\Gamma)$ with Grassmann parity $\varepsilon(\Theta^a)=\varepsilon_a$. (In this Subsection, the defining set of constraints is kept fixed for simplicity. We will consider reparametrizations of the

constraints in Subsection 3.4.) Assume that the Θ^a constraints are second-class in the antibracket sense, *i.e.* the antibracket matrix

$$E^{ab} \equiv (\Theta^a, \Theta^b) \tag{3.1}$$

of the Θ^a constraints has by definition an inverse matrix E_{ab},

$$E_{ab}E^{bc} = \delta^c_a \ . \tag{3.2}$$

The Dirac antibracket is defined completely analogous to the usual Dirac bracket for even Poisson brackets [6],

$$(F,G)_D \equiv (F,G) - (F,\Theta^a)E_{ab}(\Theta^b,G) \ , \tag{3.3}$$

or in coordinates,

$$E^{AB}_{(D)} \equiv E^{AB} - (\Gamma^A,\Theta^a)E_{ab}(\Theta^b,\Gamma^B) \ . \tag{3.4}$$

The Dirac antibracket satisfies a strong Jacobi identity

$$\sum_{F,G,H \text{ cycl.}} (-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}((F,G)_D,H)_D = 0 \ . \tag{3.5}$$

The adjective "strong" stresses the fact that the Jacobi identity holds off-shell with respect to the second-class constraints Θ^a, *i.e.* everywhere in the phase space M. There is a canonical Dirac two-form given by

$$E^D \equiv E - \frac{1}{2}d\Theta^a E_{ab} \wedge d\Theta^b \ , \tag{3.6}$$

or in local coordinates

$$E^{(D)}_{AB} \equiv E_{AB} - (\overrightarrow{\partial^l_A}\Theta^a)E_{ab}(\Theta^b\overleftarrow{\partial^r_B}) \ . \tag{3.7}$$

The two-form field $E^{(D)}_{AB}$ is compatible with the Dirac bracket, *i.e.* it satisfies the property (2.4), but it is *not* necessarily closed. Local coordinates $\Gamma^A = \{\gamma^A;\Theta^a\}$, where the second-class constraints Θ^a are part of the coordinates, are called *unitarizing* coordinates. In the physics terminology, the second-class constraints Θ^a represent unphysical degrees of freedom, which can be eliminated from the system, *i.e.* put to zero, to reveal a reduced submanifold $\tilde{M}$, whose coordinates γ^A constitute the true physical degrees of freedom. *Notation*: We use capital roman letters A, B, C, ... from the beginning of the alphabet as upper index for both the full and the reduced variables Γ^A and γ^A, respectively. A tilde "$\sim$" over an object will denote the corresponding reduced object.

Unitarizing coordinates $\Gamma^A = \{\gamma^A;\Theta^a\}$, where the second-class variables Θ^a and the physical variables γ^A are perpendicular to each other in the antibracket sense

$$(\gamma^A,\Theta^a) = 0 \ , \tag{3.8}$$

are called *transversal* coordinates. One may prove that transversal coordinate systems exist locally, although one might have to reparametrize the Θ^a constraints in order to get to them, cf. Subsection 3.4 below.

3.2 The Dirac Operators Δ_{E_D} and Δ_{ρ,E_D}

The next step is to build Khudaverdian's BV operator Δ_{E_D} for the degenerate Dirac antibracket structure (3.3), and, if a density ρ is available, the odd Laplacian Δ_{ρ,E_D}. In other words, one should substitute $E \to E_D$ everywhere in the previous Section 2. Some facts about the Δ_{E_D} operator are

immediately clear. First of all, it is covariant under general coordinate transformations, cf. Subsection 2.4. Furthermore, it is strongly nilpotent

$$\Delta_{E_D}^2 \;=\; 0 \;, \tag{3.9}$$

due to the strong Jacobi identity (3.5) and Theorem 2.10. The following Proposition 3.1 expresses the Dirac odd scalar ν_{ρ,E_D} in terms of the non-degenerate antisymplectic structure and the second-class constraints Θ^a.

Proposition 3.1 *The* **Dirac odd scalar** ν_{ρ,E_D} *is given by*

$$\nu_{\rho,E_D} \;=\; \nu_\rho - \frac{\nu_{\rho,D}^{(6)}}{2} - \frac{\nu_{\rho,D}^{(7)}}{2} - \frac{\nu_D^{(8)}}{8} + \frac{\nu_D^{(9)}}{24} \;, \tag{3.10}$$

where $\nu_\rho \equiv \nu_{\rho,E}$ *is the odd scalar for the non-degenerate antisymplectic structure* E*, and*

$$\begin{aligned}
\nu_{\rho,D}^{(6)} &\equiv (\Delta_\rho\Theta^a)E_{ab}(\Delta_\rho\Theta^b)(-1)^{\varepsilon_b} \;, & (3.11)\\
\nu_{\rho,D}^{(7)} &\equiv (-1)^{\varepsilon_a+\varepsilon_b}(\Theta^a, E_{ab}(\Delta_\rho\Theta^b)) \;=\; (\Theta^a, (\Delta_\rho\Theta^b)E_{ba}) \;, & (3.12)\\
\nu_D^{(8)} &\equiv (-1)^{\varepsilon_b}(\Theta^a, (\Theta^b, E_{ba})) \;, & (3.13)\\
\nu_D^{(9)} &\equiv (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}(\Theta^d, E_{ab})E^{bc}(E_{cd}, \Theta^a)\\
&= -(-1)^{\varepsilon_b}(\Theta^a, E^{bc})E_{cd}(\Theta^d, E_{ba}) \;. & (3.14)
\end{aligned}$$

Proof of Proposition 3.1: Since both sides of eq. (3.10) are scalars under general coordinate transformations, it is sufficient to work in Darboux coordinates for the non-degenerate E^{AB} structure. By straightforward calculation, one gets

$$\begin{aligned}
\nu_{\rho,D}^{(0)} &= \nu_\rho^{(0)} - (\Delta_1\Theta^a)E_{ab}(\Theta^b, \ln\sqrt{\rho}) - \frac{(-1)^{\varepsilon_a}}{2\sqrt{\rho}}(\Theta^a, E_{ab}(\Theta^b, \sqrt{\rho})) \;, & (3.15)\\
\nu_D^{(1)} &= -4(\Delta_1\Theta^a)E_{ab}(\Delta_1\Theta^b)(-1)^{\varepsilon_b} - 4(-1)^{\varepsilon_a+\varepsilon_b}(\Theta^a, E_{ab}(\Delta_1\Theta^b))\\
&\quad -\nu_D^{(8)} - (-1)^{\varepsilon_a}(\overrightarrow{\partial^l_A}\Theta^a,\; E_{ab})(\Theta^b, \Gamma^A) \;, & (3.16)\\
\nu_D^{(2)} &= (-1)^{\varepsilon_b\varepsilon_c}E_{ca}(\Theta^a,\; \Theta^b\overleftarrow{\partial^r_B})E^{BC}_{(D)}(\overrightarrow{\partial^l_C}\Theta^c,\; \Theta^d)E_{db}\\
&= -(-1)^{\varepsilon_a}(\Theta^b,\; \Theta^a\overleftarrow{\partial^r_A})(\Gamma^A, E_{ab})_D \;=\; -(-1)^{\varepsilon_a}(\Theta^b,\; \Theta^a\overleftarrow{\partial^r_A})(\Gamma^A, E_{ab}) - \frac{\nu_D^{(9)}}{3} \;, & (3.17)\\
\nu_D^{(3)} &= 0 \;, & (3.18)\\
\nu_D^{(4)} &= 0 \;, & (3.19)\\
\nu_D^{(5)} &= 0 \;. & (3.20)
\end{aligned}$$

The pertinent linear combination (2.26) of eqs. (3.15)–(3.20) yields the eq. (3.10).

□

3.3 Annihilation Relations

The fact that the Θ^a constraints are null-directions for the Dirac construction is reflected slightly differently in 1) the Dirac antibracket $(\cdot,\cdot)_D$, 2) the Dirac odd Laplacian Δ_{ρ,E_D}, and 3) the Δ_{E_D}

operator. Explicitly, for a scalar function F, a density ρ and a semidensity σ, one has

$$
\begin{aligned}
(F,\Theta^{a})_{D} &= 0 \,, && (3.21)\\
(\Delta_{\rho,E_{D}}\Theta^{a}) &= \frac{(-1)^{\varepsilon_{A}}}{2\rho}\overrightarrow{\partial^{l}_{A}}\rho(\Gamma^{A},\Theta^{a})_{D} \;=\; 0 \,, && (3.22)\\
[\overrightarrow{\Delta}_{E_{D}},\Theta^{a}]\sigma &= [\overrightarrow{\Delta}_{1,E_{D}},\Theta^{a}]\sigma \;=\; (\Delta_{1,E_{D}}\Theta^{a})\sigma+(-1)^{\varepsilon_{a}}(\Theta^{a},\sigma)_{D} \;=\; 0 \,, && (3.23)
\end{aligned}
$$

respectively. Eqs. (3.21)–(3.23) generalize to

$$
\begin{aligned}
(F,f(\Theta))_{D} &= 0 \,, && (3.24)\\
(\Delta_{\rho,E_{D}}f(\Theta)) &= 0 \,, && (3.25)\\
[\overrightarrow{\Delta}_{E_{D}},f(\Theta)]\sigma &= 0 \,, && (3.26)
\end{aligned}
$$

for an arbitrary function $f(\Theta)$ of the constraints Θ^{a}. (In other words: f is here assumed not to depend on the physical variables γ^{A}.) Note however, that if Θ^{a} is not among the defining set of constraints, but only a linear combination of those (*i.e.* the coefficients in the linear combination could involve the physical variables γ^{A}), the last equality in each of the above eqs. (3.21)–(3.26) becomes weak, *i.e.* there could be off-shell contributions, cf. next Subsection 3.4 and Ref. [8].

3.4 Reparametrization of Second-Class Constraints

A general and tricky feature of the Dirac construction, is, that it *changes* if one uses another defining set of second-class constraints

$$
\Theta^{a} \qquad\longrightarrow\qquad \Theta'^{a} \;=\; \Lambda^{a}{}_{b}(\Gamma)\;\Theta^{b} \,. \tag{3.27}
$$

However, the dependence is so soft that physics, which lives on-shell, is not affected [8]. We shall here clarify in exactly what sense the $\Delta_{E_{D}}$ operator remains invariant on-shell under reparametrization of the constraints.

To warm up, let us recall that the Dirac antibrackets $(F,G)_{D}$ and $(F,G)'_{D}$, defined using the primed and unprimed constraints Θ'^{a} and Θ^{a}, respectively, are the same on-shell

$$
(F,G)'_{D} \;\approx\; (F,G)_{D} \,. \tag{3.28}
$$

Here the symbol "$\approx$" is the Dirac weak equivalence symbol, which denotes equivalence modulo terms of order $\mathcal{O}(\Theta)$. More generally,

$$
F' \;\approx\; F \;\;\wedge\;\; G' \;\approx\; G \qquad\Rightarrow\qquad (F',G')'_{D} \;\approx\; (F,G)_{D} \,. \tag{3.29}
$$

Hence the reduced bracket

$$
(\tilde{F},\tilde{G})_{\sim} \;\equiv\; (F,G)_{D}|_{\Theta=0} \,, \tag{3.30}
$$

is independent of both the choice of constraints Θ^{a} and the representatives $F=F(\Gamma)$, $G=G(\Gamma)$ on M. Here $\tilde{F}\equiv F|_{\Theta=0}=\tilde{F}(\gamma)$ and $\tilde{G}\equiv G|_{\Theta=0}=\tilde{G}(\gamma)$ are functions on the physical submanifold $\tilde{M}$.

On the other hand, to have a well-defined notion of reduced densities and semidensities on the physical submanifold $\tilde{M}$, it is necessary to let the densities and semidensities transform as

$$
\rho' \;\approx\; \rho\,\Lambda \,, \qquad\qquad \sigma' \;\approx\; \sigma\,\sqrt{\Lambda} \,, \tag{3.31}
$$

under reparametrization of the defining set of constraints $\Theta^{a} \to \Theta'^{a} = \Lambda^{a}{}_{b}\, \Theta^{b}$. Here

$$\Lambda \;\equiv\; \mathrm{sdet}(\Lambda^{a}{}_{b}) \tag{3.32}$$

denotes the superdeterminant of the reparametrization matrix $\Lambda^{a}{}_{b} = \Lambda^{a}{}_{b}(\Gamma)$. The reduction

$$\tilde{\rho} \;\equiv\; \rho|_{\Theta=0} \;, \qquad\qquad \tilde{\sigma} \;\equiv\; \sigma|_{\Theta=0} \;, \tag{3.33}$$

is then by definition performed in a unitarizing coordinate system $\Gamma^{A} = \{\gamma^{A}; \Theta^{a}\}$, where it is implicitly understood that the Θ^{a} coordinates coincide with the defining set of constraints. Similar to the Dirac antibracket $(\cdot,\cdot)_{D}$, we imagine that the densities and semidensities refer to an internal defining set of Θ^{a} constraints. If one chooses another defining set of constraints Θ'^{a}, and an accompanying unitarizing coordinate system $\Gamma'^{A} = \{\gamma'^{A}; \Theta'^{a}\}$, the superdeterminant factor Λ in the reparametrization rule (3.31) is designed to cancel the Jacobian factor J from the coordinate transformation (2.14) on-shell, so that the reduced definition (3.33) stays the same.

Similarly, it is necessary that the $\Delta_{E_{D}}$ operator, which takes semidensities to semidensities, transforms as

$$\Delta'_{E_{D}} \;\approx\; \sqrt{\Lambda}\;\Delta_{E_{D}}\frac{1}{\sqrt{\Lambda}} \tag{3.34}$$

as an operator identity. Stated more precisely, the odd scalar $\nu_{\rho,E_{D}}$ from Definition 2.8 should be invariant on-shell

$$\nu_{\rho',E'_{D}} \;\approx\; \nu_{\rho,E_{D}} \tag{3.35}$$

under reparametrization of the constraints. This is the core issue at stake. To prove that it indeed holds, first note that it is enough to check the claim (3.35) if the set of unprimed constraints Θ^{a} happens to belong to a set of transversal coordinates $\Gamma^{A} = \{\gamma^{A}; \Theta^{a}\}$. (If this is not the case, one can always locally find a transversal coordinate system, and split the above reparamerization into two successive reparamerizations that both involve the transversal coordinates.) Transversal coordinates will simplify considerably the ensuing calculations. In general, the on-shell change of $\nu_{\rho,E_{D}}$ depends on how the Dirac antibracket $(\cdot,\cdot)_{D}$ changes up to the second order in Θ^{a}, cf. eq. (4.39) in Ref. [8]. Explicitly, one may show that the quantities $\nu^{(0)}_{\rho,D}$, $\nu^{(1)}_{D}$, $\nu^{(2)}_{D}$, $\nu^{(3)}_{D}$, $\nu^{(4)}_{D}$ and $\nu^{(5)}_{D}$, defined in eqs. (2.21)–(2.25) and (2.27), transform as

$$\nu'^{(0)}_{\rho,D} \;\equiv\; \frac{1}{\sqrt{\rho'}}(\Delta_{1,E'_{D}}\sqrt{\rho'}) \;\approx\; \frac{1}{\sqrt{\Lambda\rho}}(\Delta_{\frac{1}{\Lambda},E_{D}}\sqrt{\Lambda\rho}) \;=\; \nu^{(0)}_{\rho,D} - \sqrt{\Lambda}(\Delta_{1,E_{D}}\frac{1}{\sqrt{\Lambda}}) \;, \tag{3.36}$$

$$\nu'^{(1)}_{D} \;\approx\; \nu^{(1)}_{D} + 8\sqrt{\Lambda}(\Delta_{1,E_{D}}\frac{1}{\sqrt{\Lambda}}) - (-1)^{\varepsilon_{b}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Theta'^{a}}\Theta^{b},\; \frac{\overrightarrow{\partial^{l}}}{\partial\Theta^{b}}\Theta'^{a})_{D} \;, \tag{3.37}$$

$$\nu'^{(2)}_{D} \;\approx\; \nu^{(2)}_{D} - (-1)^{\varepsilon_{b}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Theta'^{a}}\Theta^{b},\; \frac{\overrightarrow{\partial^{l}}}{\partial\Theta^{b}}\Theta'^{a})_{D} \;, \tag{3.38}$$

$$\nu'^{(3)}_{D} \;\approx\; \nu^{(3)}_{D} \;, \tag{3.39}$$

$$\nu'^{(4)}_{D} \;\approx\; \nu^{(4)}_{D} \;, \tag{3.40}$$

$$\nu'^{(5)}_{D} \;\approx\; \nu^{(5)}_{D} \;. \tag{3.41}$$

The last equality in eq. (3.36) is a non-trivial property of the odd Laplacian. It is now easy to see that the relevant linear combination $\nu_{\rho,E_{D}}$ of $\nu^{(0)}_{\rho,D}$, $\nu^{(1)}_{D}$, $\nu^{(2)}_{D}$, $\nu^{(3)}_{D}$, $\nu^{(4)}_{D}$ and $\nu^{(5)}_{D}$ is invariant on-shell.

3.5 Nilpotency Condition for the odd Dirac Laplacian Δ_{ρ,E_D}

One of the surprising conclusions of Ref. [8] was that one cannot maintain a strong nilpotency of the Dirac odd Laplacian Δ_{ρ,E_D} under reparametrization of the second-class constraints. This is consistent with our new results. Using the terminology of last Subsection 3.4, one would say that the effect is caused by the off-shell variations of the odd scalar ν_{ρ,E_D} and the Dirac antibracket $(\cdot,\cdot)_D$, cf. the following calculation:

$$\Delta^2_{\rho',E'_D} \;=\; (\nu_{\rho',E'_D},\;\cdot\;)'_D \;\approx\; (\nu_{\rho,E_D},\;\cdot\;)_D \;=\; \Delta^2_{\rho,E_D}\;. \tag{3.42}$$

Here use is made of eqs. (2.43), (3.29) and (3.35). This should be compared to the situation with the Δ_{E_D} operator where the strong nilpotency (3.9) is manifest from the onset, regardless of which defining set of Θ^a constraints is used.

3.6 Dirac Partition Function

As an application of the Δ_{E_D} operator, it is interesting to consider the first-level Dirac partition function in the $\lambda^*_\alpha\!=\!0$ gauge. A review of the first-level formalism can be found in Ref. [8]. The partition function reads

$$\mathcal{Z}_D \;=\; \int [d\Gamma][d\lambda]\,\exp[\frac{i}{\hbar}(W_{E_D}+X_{E_D})]\Big|_{\lambda^*=0}\prod_a\delta(\Theta^a)\;, \tag{3.43}$$

where $W_{E_D}\!=\!W_{E_D}(\Gamma)$ and $X_{E_D}\!=\!X_{E_D}(\Gamma;\lambda,\lambda^*)$ satisfy the Quantum Master Equations

$$\Delta_{E_D}\exp[\frac{i}{\hbar}W_{E_D}] \;=\; 0\;, \tag{3.44}$$

$$((-1)^{\varepsilon_\alpha}\frac{\overrightarrow{\partial^l}}{\partial\lambda^\alpha}\frac{\overrightarrow{\partial^l}}{\partial\lambda^*_\alpha}+\Delta_{E_D})\exp[\frac{i}{\hbar}X_{E_D}] \;=\; 0\;. \tag{3.45}$$

The formula (3.43) for the Dirac partition function $\mathcal{Z}_D$ differs from the original formula [8, 14] by not depending on a ρ. Instead, the partition function $\mathcal{Z}_D$ is invariant under general coordinate transformations and under reparametrization of the Θ^a constraints because the Boltzmann semidensities $\exp[\frac{i}{\hbar}W_{E_D}]$ and $\exp[\frac{i}{\hbar}X_{E_D}]$ transform according to (2.14) and (3.31). Given an arbitrary density ρ, it is possible to introduce Boltzmann scalars

$$\exp[\frac{i}{\hbar}W_\rho] \;\equiv\; \exp[\frac{i}{\hbar}W_{E_D}]/\sqrt{\rho}\;, \tag{3.46}$$

$$\exp[\frac{i}{\hbar}X_\rho] \;\equiv\; \exp[\frac{i}{\hbar}X_{E_D}]/\sqrt{\rho}\;, \tag{3.47}$$

which satisfy corresponding Modified Quantum Master Equations similar to eq. (1.4).

4 Conversion of Second-Class into First-Class

Originally, the conversion of second-class constraints into first-class constraints was developed for even Poisson geometry [15, 16, 17, 18]. Later it was adapted to anti-Poisson geometry in Ref. [19], more precisely to the Dirac antibracket $(\cdot,\cdot)_D$ and odd Laplacian Δ_{ρ,E_D}. In this Section 4 we develop the anti-Poisson conversion method further and show that the Dirac Δ_{E_D} operator from last Section 3 can also be derived via conversion.

4.1 Extended Manifold $M_{\rm ext}$

As in Section 3 the starting point is a general non-degenerate antisymplectic manifold $(M;E)$ with a set of globally defined second-class constraints $\Theta^a = \Theta^a(\Gamma)$, which have Grassmann parity $\varepsilon(\Theta^a)\!=\!\varepsilon_a$. We now consider a cartesian product $M_{\rm ext} \equiv M \times V$, where $(V;\omega)$ is a vector space with a constant and non-degenerate antisymplectic metric, and such that the dimension of V is equal to the number of Θ^a constraints. We will often identify M with $M \times \{0\} \subseteq M_{\rm ext}$. The extended manifold $M_{\rm ext}$ has antisymplectic structure $E_{\rm ext} \equiv E \oplus \omega$.

Assume that points (*i.e.* vectors) in the vector space V are described by a set of coordinates Φ_a with Grassmann parity $\varepsilon(\Phi_a)\!=\!\varepsilon_a\!+\!1$. For each set of local coordinates Γ^A for the manifold M, the extended manifold $M_{\rm ext}$ will have local coordinates $\Gamma^A_{\rm ext}\!\equiv\!\{\Gamma^A;\Phi_a\}$. *Notation*: We use capital roman letters A, B, C, $\ldots$ from the beginning of the alphabet as upper index for both the original and the extended variables Γ^A and $\Gamma^A_{\rm ext}$, respectively. In detail, the extended antibracket $(\cdot,\cdot)_{\rm ext}$ on $M_{\rm ext}$ reads

$$(\Gamma^A,\Gamma^B)_{\rm ext} \;\equiv\; (\Gamma^A,\Gamma^B) \;=\; E^{AB} \;, \tag{4.1}$$

$$(\Gamma^A,\Phi_a)_{\rm ext} \;\equiv\; 0 \;, \tag{4.2}$$

$$(\Phi_a,\Phi_b)_{\rm ext} \;\equiv\; \omega_{ab} \;, \qquad\qquad \varepsilon(\omega_{ab}) \;=\; \varepsilon_a\!+\!\varepsilon_b\!+\!1 \;, \tag{4.3}$$

where, in particular, the antisymplectic matrix $\omega_{ab} = -(-1)^{\varepsilon_a\varepsilon_b}\omega_{ba}$ does not depend on Γ^A nor on Φ_a. In other words, up to a constant matrix, the Φ_a coordinates are global Darboux coordinates for the vector space V.

4.2 First-Class Constraints T^a

One next seeks Abelian first-class constraints $T^a\!=\!T^a(\Gamma;\Phi)$ such that

$$(T^a,T^b)_{\rm ext} \;=\; 0 \;, \qquad\qquad T^a|_{\Phi=0} \;=\; \Theta^a \;. \tag{4.4}$$

Eq. (4.4) is the defining relation for the conversion of second-class constraints Θ^a into first-class constraint T^a. The first-class constraints T^a are treated as power series expansions in the Φ_a variables

$$T^a \;=\; \Theta^a + \left\{ \begin{array}{c} X_L^{ab}\Phi_b \\ \Phi_b X_R^{ba} \end{array} \right\} + \frac{1}{2}\left\{ \begin{array}{c} Y_L^{abc}\Phi_c\Phi_b \\ \Phi_b Y_M^{bac}\Phi_c \\ \Phi_b\Phi_c Y_R^{cba} \end{array} \right\} + \frac{1}{6}Z_L^{abcd}\Phi_d\Phi_c\Phi_b + {\cal O}(\Phi^4) \;. \tag{4.5}$$

The expressions $X_L^{ab}\Phi_b \equiv \Phi_b X_R^{ba}$ and $Y_L^{abc}\Phi_c\Phi_b \equiv \Phi_b Y_M^{bac}\Phi_c \equiv \Phi_b\Phi_c Y_R^{cba}$ inside the curly brackets "{ }" of eq. (4.5) reflect various (equivalent) ways of ordering the Φ^a variables. The rules for shifting between the ordering prescriptions are

$$X_L^{ab} \;=\; (-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}X_R^{ba} \;, \tag{4.6}$$

$$(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}Y_L^{bac} \;=\; Y_M^{abc} \;=\; (-1)^{(\varepsilon_b+1)(\varepsilon_c+1)}Y_R^{acb} \;. \tag{4.7}$$

One may show that a solution T^a to the system (4.4) exists, but that it is not unique. For instance, the condition on the $X^{ab}\!=\!X^{ab}(\Gamma)$ structure functions reads

$$E^{ad} \;\equiv\; (\Theta^a,\Theta^d) \;=\; -\,X_L^{ab}\omega_{bc}X_R^{cd} \;. \tag{4.8}$$

The matrices X_L^{ab} and X_R^{ab} are necessarily invertible with inverse matrices $X^L_{ab}\!=\!(-1)^{\varepsilon_a\varepsilon_b}X^R_{ab}$, since both $E^{ab}\!\equiv\!(\Theta^a,\Theta^b)$ and $\omega_{ab}\!\equiv\!(\Phi_a,\Phi_b)_{\rm ext}$ in eq. (4.8) are invertible. One may view X^{ab} as a Grassmann-odd vielbein between the curved second-class matrix E^{ab} and the flat metric ω_{ab}. At the next order in Φ^a, the condition on the $Y^{abc}\!=\!Y^{abc}(\Gamma)$ structure functions reads

$$(\Theta^a,X_R^{cb}) + X_L^{ad}\omega_{de}Y_R^{ecb} + (X_L^{ac},\Theta^b) + Y_L^{acd}\omega_{de}X_R^{eb} \;=\; 0 \;, \tag{4.9}$$

and so forth.

4.3 Gauge Invariance

The idea is now to view the first-class constraints T^a as generators of gauge symmetry and $\Phi_a = 0$ as a particular gauge. We start by defining gauge-invariant observables on the extended manifold $M_{\rm ext}$.

Definition 4.1 *A scalar function* $\bar{F} = \bar{F}(\Gamma;\Phi)$*, a density* $\bar{\rho} = \bar{\rho}(\Gamma;\Phi)$ *or a semidensity* $\bar{\sigma} = \bar{\sigma}(\Gamma;\Phi)$ *on the extended manifold* $M_{\rm ext}$ *is called a* **gauge-invariant extension** *of a scalar function* $F = F(\Gamma)$*, a density* $\rho = \rho(\Gamma)$ *or a semidensity* $\sigma = \sigma(\Gamma)$ *on the original manifold* M*, if the following conditions are satisfied*

$$
\begin{aligned}
(\bar{F}, T^a)_{\rm ext} &= 0 \ , & \bar{F}|_{\Phi=0} &= F \ , & (4.10)\\
(\Delta_{\bar{\rho}} T^a) &= 0 \ , & \bar{\rho}|_{\Phi=0} &= \rho j \ , & (4.11)\\
[\overrightarrow{\Delta}_{E_{\rm ext}}, T^a]\bar{\sigma} &= 0 \ , & \bar{\sigma}|_{\Phi=0} &= \sigma\sqrt{j} \ , & (4.12)
\end{aligned}
$$

respectively, where the j*-factor is defined in eq. (4.13) below.*

4.4 The j-Factor

The factor

$$
j \ \equiv \ \bar{j}|_{\Phi=0} \ = \ {\rm sdet}(\omega_{ac} X_R^{cb}) \tag{4.13}
$$

is defined as the $\Phi = 0$ restriction of the superdeterminant

$$
\bar{j} \ \equiv \ {\rm sdet}(\Phi_a, T^b)_{\rm ext} \ = \ \int [d\bar{C}][dC] \ \exp\left[\frac{i}{\hbar}\bar{C}^a (\Phi_a, T^b)_{\rm ext} C_b\right] \ , \qquad \varepsilon(\bar{C}^a) \ = \ \varepsilon_a + 1 \ = \ \varepsilon(C_a) \ . \tag{4.14}
$$

The j-factor (4.13) is independent of the choice of X^{ab} structure functions because of eq. (4.8). It is a density for the vector space V such that the corresponding volume form $j[d\Phi]$ on V is independent of the choice of coordinates Φ_a. In this way the multiplication with j in eq. (4.11) transforms a density ρ on the manifold M into a density ρj for the extended manifold $M_{\rm ext} \equiv M \times V$. The j-factor is unique up to an overall constant and can be physically explained as a Faddeev-Popov determinant, see Subsection 4.7.

Below we shall overwhelmingly justify the j-factor in Definition 4.1, in particular, through the Conversion Theorem 4.2, but let us start by briefly mentioning a curious implication. Consider what happens to the set of vielbein solutions X_L^{ab} to eq. (4.8) under reparametrizations of the defining set of second-class constraints $\Theta^a \to \Theta'^a = \Lambda^a{}_b \, \Theta^b$. It is natural to expect that there exists a bijective map $X_L^{ab} \to X_L'^{ab}$ between the solutions such that

$$
X_L'^{ac} \ \approx \ \Lambda^a{}_b X_L^{bc} \ , \tag{4.15}
$$

where "$\approx$" denotes weak equivalence, cf. Subsection 3.4. According to such map, the j-factor would transform as

$$
j' \ \approx \ \Lambda \, j \ . \tag{4.16}
$$

Recalling the transformation rule (3.31) for ρ, this implies that the density $\bar{\rho}|_{\Phi=0} = \rho j$ on $M_{\rm ext}$ changes with the *square* of Λ,

$$
\bar{\rho}'|_{\Phi=0} \ \approx \ \Lambda^2 \, \bar{\rho}|_{\Phi=0} \ . \tag{4.17}
$$

So while the j-factor does indeed cancel the effect of changing the Φ_a coordinates, it *doubles* the effect of changing the second-class constraints Θ^a! Nevertheless, this doubling phenomenon fits nicely with the rest of the conversion construction, cf. Subsection 4.7 below.

4.5 Discussion of Gauge Invariance

Let us now justify the conditions (4.10)–(4.12). The first condition (4.10) is simply the antisymplectic definition of gauge invariance. As an example of condition (4.10), note that a first-class constraint $T^a = \bar{\Theta}^a$ is a gauge-invariant extension of the corresponding second-class constraint Θ^a. The other two conditions (4.11) and (4.12) are a priori less obvious, but there are many reasons to impose them:

1. The three conditions (4.10)–(4.12) are covariant with respect to coordinate changes.
2. The conditions (4.10)–(4.12) are consistent with each others, say, if one considers a density $\rho' = \rho F$, or a semidensity $\sigma = \sqrt{\rho}$.
3. The conditions (4.10)–(4.12) are natural counterparts of the annihilation properties (3.21)–(3.23).
4. One may show that there exist unique gauge-invariant extensions $\bar{F}$, $\bar{\rho}$ and $\bar{\sigma}$ satisfying the condition (4.10), (4.11) and (4.12), respectively.
5. The extended antibracket $(\cdot,\cdot)_{\rm ext}$, the extended odd Laplacian $\Delta_{\bar{\rho}} \equiv \Delta_{\bar{\rho},E_{\rm ext}}$, the extended $\Delta_{E_{\rm ext}}$ operator and the extended odd scalar $\nu_{\bar{\rho}} \equiv \nu_{\bar{\rho},E_{\rm ext}}$ are compatibly with the gauge-invariance conditions (4.10)–(4.12), *i.e.*

$$\begin{aligned}
(\bar{F}\bar{G},T^a)_{\rm ext} &= \bar{F}(\bar{G},T^a)_{\rm ext} + (-1)^{\varepsilon_F\varepsilon_G}\bar{G}(\bar{F},T^a)_{\rm ext} \;=\; 0 \;, && (4.18)\\
((\bar{F},\bar{G})_{\rm ext},T^a)_{\rm ext} &= (\bar{F},(\bar{G},T^a)_{\rm ext})_{\rm ext} + (-1)^{(\varepsilon_a+1)(\varepsilon_G+1)}((\bar{F},T^a)_{\rm ext},\bar{G})_{\rm ext} \;=\; 0 \;, && (4.19)\\
(\Delta_{\bar{\rho}}\bar{F},T^a)_{\rm ext} &= \Delta_{\bar{\rho}}(\bar{F},T^a)_{\rm ext} + (-1)^{\varepsilon_F}(\bar{F},\Delta_{\bar{\rho}}T^a)_{\rm ext} \;=\; 0 \;, && (4.20)\\
[\overrightarrow{\Delta}_{E_{\rm ext}},T^a](\Delta_{E_{\rm ext}}\bar{\sigma}) &= (\Delta_{E_{\rm ext}}T^a\Delta_{E_{\rm ext}}\bar{\sigma}) \;=\; (\Delta_{E_{\rm ext}}[T^a,\overrightarrow{\Delta}_{E_{\rm ext}}]\bar{\sigma}) \;=\; 0 \;, && (4.21)\\
(\nu_{\bar{\rho}},T^a)_{\rm ext} &= (\Delta_{\bar{\rho}}^2 T^a) \;=\; 0 \;. && (4.22)
\end{aligned}$$

   Here use is made of the ordinary Leibniz rule, the Jacobi identity (2.2), the BV Leibniz rule (2.16), the eq. (2.19) and the eq. (2.43), respectively.
6. The conditions (4.10)–(4.12) imply the Conversion Theorem 4.2 below.

4.6 The Conversion Map

The gauge-invariant extension map

$$\mathcal{F}(M) \;\ni\; F \;\stackrel{\cong}{\longmapsto}\; \bar{F} \;\in\; \mathcal{F}(M_{\rm ext})_{\rm inv} \tag{4.23}$$

(which is also known as the *conversion map*) is an isomorphism of functions on M to gauge-invariant function on $M_{\rm ext}$, cf. point 4 of the last Subsection 4.5. The inverse conversion map is simply the restriction to M,

$$\mathcal{F}(M_{\rm ext})_{\rm inv} \;\ni\; \bar{F} \;\stackrel{\cong}{\longmapsto}\; \bar{F}\big|_{\Phi=0} \;\in\; \mathcal{F}(M) \;. \tag{4.24}$$

The following Theorem 4.2 is the heart of the conversion method. It shows that the inverse conversion map transforms the extended model into the Dirac construction.

Theorem 4.2 *The restrictions to M of the extended antibracket $(\cdot,\cdot)_{\rm ext}$, the extended odd Laplacian $\Delta_{\bar{\rho}} \equiv \Delta_{\bar{\rho},E_{\rm ext}}$, the extended $\Delta_{E_{\rm ext}}$ operator and the extended odd scalar $\nu_{\bar{\rho}} \equiv \nu_{\bar{\rho},E_{\rm ext}}$ reproduce the*

corresponding Dirac constructions:

$$
\begin{aligned}
(\bar{F},\bar{G})_{\rm ext}\big|_{\Phi=0} &= (F,G)_{D} \ , & (4.25)\\
(\Delta_{\bar{\rho},E_{\rm ext}}\bar{F})\Big|_{\Phi=0} &= (\Delta_{\rho,E_{D}}F) \ , & (4.26)\\
(\Delta_{E_{\rm ext}}\bar{\sigma})\Big|_{\Phi=0} &= \sqrt{j}(\Delta_{E_{D}}\sigma) \ , & (4.27)\\
\nu_{\bar{\rho},E_{\rm ext}}\Big|_{\Phi=0} &= \nu_{\rho,E_{D}} \ . & (4.28)
\end{aligned}
$$

In principle, it is enough to prove eq. (4.28), since eq. (4.28) $\Leftrightarrow$ eq. (4.27) $\Rightarrow$ eq. (4.26) $\Rightarrow$ eq. (4.25). Nevertheless, we shall give independent proofs of eqs. (4.25), (4.26) and (4.28) in Appendix C. The following Corollary 4.3 restates the conclusions of Conversion Theorem 4.2 using the forward conversion map.

Corollary 4.3

$$
\begin{aligned}
(FG)^{-} &= \bar{F}\bar{G} \ , & (4.29)\\
((F,G)_{D})^{-} &= (\bar{F},\bar{G})_{\rm ext} \ , & (4.30)\\
(\Delta_{\rho,E_{D}}F)^{-} &= (\Delta_{\bar{\rho},E_{\rm ext}}\bar{F}) \ , & (4.31)\\
(\sqrt{j}\Delta_{E_{D}}\sigma)^{-} &= (\Delta_{E_{\rm ext}}\bar{\sigma}) \ , & (4.32)\\
(\nu_{\rho,E_{D}})^{-} &= \nu_{\bar{\rho},E_{\rm ext}} \ . & (4.33)
\end{aligned}
$$

In particular, eqs. (4.25) and (4.30) show that the conversion map is an isomorphism in the sense of anti-Poisson algebras between the Dirac anti-Poisson algebra $(\mathcal{F}(M);(\cdot,\cdot)_{D})$ and the anti-Poisson algebra $(\mathcal{F}(M_{\rm ext})_{\rm inv};(\cdot,\cdot)_{\rm ext})$ of gauge-invariant functions on $M_{\rm ext}$.

4.7 Extended Partition Function

The first-level partition function in the $\lambda^{*}_{\alpha}=0$ gauge reads

$$
\mathcal{Z}_{\rm ext} \ = \ \int[d\Gamma_{\rm ext}][d\lambda]\ \exp[\frac{i}{\hbar}(W_{E_{\rm ext}}+X_{E_{\rm ext}})]\Big|_{\lambda^{*}=0}\frac{1}{{\rm sdet}(\chi_{a},T^{b})_{\rm ext}}\prod_{c}\delta(T^{c})\prod_{d}\delta(\chi_{d}) \ , \qquad (4.34)
$$

where $W_{E_{\rm ext}}\!=\!W_{E_{\rm ext}}(\Gamma_{\rm ext})$ and $X_{E_{\rm ext}}\!=\!X_{E_{\rm ext}}(\Gamma_{\rm ext};\lambda,\lambda^{*})$ satisfy the Quantum Master Equations

$$
\begin{aligned}
\Delta_{E_{\rm ext}}\exp[\frac{i}{\hbar}W_{E_{\rm ext}}] &= 0 \ , & (4.35)\\
((-1)^{\varepsilon_{\alpha}}\frac{\overrightarrow{\partial^{l}}}{\partial\lambda^{\alpha}}\frac{\overrightarrow{\partial^{l}}}{\partial\lambda^{*}_{\alpha}}+\Delta_{E_{\rm ext}})\exp[\frac{i}{\hbar}X_{E_{\rm ext}}] &= 0 \ , & (4.36)
\end{aligned}
$$

and they are gauge invariant in the sense of condition (4.12):

$$
\begin{aligned}
[\overrightarrow{\Delta}_{E_{\rm ext}},T^{a}]\exp[\frac{i}{\hbar}W_{E_{\rm ext}}] &= 0 \ , & \exp[\tfrac{i}{\hbar}W_{E_{\rm ext}}]\Big|_{\Phi=0} &= \sqrt{j}\exp[\frac{i}{\hbar}W_{E_{D}}] \ , & (4.37)\\
[\overrightarrow{\Delta}_{E_{\rm ext}},T^{a}]\exp[\frac{i}{\hbar}X_{E_{\rm ext}}] &= 0 \ , & \exp[\tfrac{i}{\hbar}X_{E_{\rm ext}}]\Big|_{\Phi=0} &= \sqrt{j}\exp[\frac{i}{\hbar}X_{E_{D}}] \ . & (4.38)
\end{aligned}
$$

Here the Boltzmann semidensities $\exp[\frac{i}{\hbar}W_{E_D}]$ and $\exp[\frac{i}{\hbar}X_{E_D}]$ satisfy the Quantum Master Equations (3.44) and (3.45), respectively. It is an important fact that in the gauge $\chi_a\!=\!\Phi_a$, the expression (4.34) for the extended partition function reduces to the Dirac partition function (3.43), *i.e.*

$$\mathcal{Z}_{\rm ext} \;=\; \mathcal{Z}_D \;. \tag{4.39}$$

Given a density $\rho\!=\!\rho(\Gamma)$ on M, and a density $\bar{\rho}\!=\!\bar{\rho}(\Gamma_{\rm ext})$ on $M_{\rm ext}$ that satisfies eq. (4.11), it is possible to introduce Boltzmann scalars

$$\begin{aligned}\exp[\frac{i}{\hbar}W_{\bar{\rho}}] &\equiv \exp[\frac{i}{\hbar}W_{E_{\rm ext}}]/\sqrt{\bar{\rho}}\;, &(4.40)\\ \exp[\frac{i}{\hbar}X_{\bar{\rho}}] &\equiv \exp[\frac{i}{\hbar}X_{E_{\rm ext}}]/\sqrt{\bar{\rho}}\;, &(4.41)\end{aligned}$$

which satisfy corresponding Modified Quantum Master Equations similar to eq. (1.4). The Quantum Actions $W_{\bar{\rho}}$ and $X_{\bar{\rho}}$ defined this way are automatically gauge invariant

$$\begin{aligned}(W_{\bar{\rho}},T^a)_{\rm ext} &= 0\;, \qquad & W_{\bar{\rho}}\Big|_{\Phi=0} &= W_{\rho}\;, &(4.42)\\ (X_{\bar{\rho}},T^a)_{\rm ext} &= 0\;, \qquad & X_{\bar{\rho}}\Big|_{\Phi=0} &= X_{\rho}\;. &(4.43)\end{aligned}$$

Here W_ρ and X_ρ are defined in eqs. (3.46) and (3.47), respectively.

5 Conclusions

We have shown for a general degenerate anti-Poisson manifold (under the relatively mild assumption of a compatible two-form field) how to define in arbitrary coordinates the Δ_E operator, which takes semidensities to semidensities, cf. Lemma 2.6. A large class of such degenerate antibrackets are provided by the Dirac antibracket construction. We have given a formula for the Dirac Δ_{E_D} operator, cf. Proposition 3.1, and shown in Subsection 3.4 that it is on-shell invariant under reparametrizations of the second-class constraints. Finally, we showed that the Dirac Δ_{E_D} operator also follows from the antisymplectic conversion scheme, cf. Conversion Theorem 4.2.

Let us conclude with the following remark. It is often pointed out that the antibracket $(\cdot,\cdot)$ is a descendant of the odd Laplacian Δ_ρ. It measures the failure of the odd Laplacian Δ_ρ to act as a linear derivation, *i.e.* to satisfy the Leibniz rule. It can be written as a double-commutator [7, 20, 21]

$$(F,G) \;=\; (-1)^{\varepsilon_F}[[\overrightarrow{\Delta}_\rho,F],G]1\;. \tag{5.1}$$

In turn, the odd Laplacian Δ_ρ is a descendant of the Δ_E operator [8, 9]

$$(\Delta_\rho F) \;=\; \frac{1}{\sqrt{\rho}}[\overrightarrow{\Delta}_E,F]\sqrt{\rho}\;. \tag{5.2}$$

That is, one has schematically the following hierarchy:

$$\begin{array}{ccc} \Delta_E \text{ operator} & & \\ \Downarrow & & \\ \text{Odd Laplacian } \Delta_\rho & \Leftarrow & \text{Density } \rho \\ \Downarrow & & \\ \text{Antibracket } (\cdot,\cdot) & & \end{array} \tag{5.3}$$

Whereas the Δ_E operator is manifestly nilpotent, cf. Theorem 2.10, there is no fundamental reason to require the odd Laplacian Δ_ρ to be nilpotent. (Of course, if Δ_ρ is not nilpotent, the Boltzmann scalar $\exp[\frac{i}{\hbar}W_\rho]$ would in general have to satisfy a Modified Quantum Master Equation with a non-trivial ν_ρ term, cf. eq. (1.4). See also the recent preprint [22].) The Dirac odd Laplacian Δ_{ρ,E_D} offers more evidence that nilpotency of the odd Laplacian is not fundamental, at least not in its strong formulation, since in this case the nilpotency can only be maintained weakly under reparametrizations of the second-class constraints Θ^a, cf. Ref. [8] and Subsection 3.5.

ACKNOWLEDGEMENT: The Author thanks I.A. Batalin and P.H. Damgaard for discussions. The Author also thanks the Erwin Schrödinger Institute for warm hospitality. This work is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409.

A Proof of bi-Darboux Theorem 2.1

If there exists an atlas of bi-Darboux coordinates, the two-form $E = d\phi^*_\alpha \wedge d\phi^\alpha$ is obviously closed. Now consider the other direction. Assume that the two-form E is closed. Then there locally exists a pre-antisymplectic one-form potential ϑ such that

$$d\vartheta \;=\; E \; . \tag{A.1}$$

Independently one knows that locally there exist Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha; \Theta^a\}$. Since the two-form E is assumed to be compatible with the anti-Poisson structure, it must be of the form (2.10). It is always possible to organize the pre-antisymplectic one-form potential as

$$\vartheta \;\sim\; \phi^*_\alpha d\phi^\alpha + \vartheta_A d\gamma^A + \vartheta'_a d\Theta^a \; , \tag{A.2}$$

where $\gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$ collectively denotes the fields and the antifields without the Casimirs. The symbol "$\sim$" denotes equality modulo exact terms, whose precise expressions are irrelevant, since we are ultimately only interested in the two-form E. It follows from eqs. (2.10), (A.1) and (A.2) that

$$(\frac{\overrightarrow{\partial^l}}{\partial\gamma^A}\vartheta_B) \;=\; (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) \; , \tag{A.3}$$

and hence there locally exists a fermionic function Ψ' such that

$$\vartheta_A \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\gamma^A}\Psi') \; . \tag{A.4}$$

Defining

$$\vartheta_a \;\equiv\; \vartheta'_a - (\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\Psi') \; , \tag{A.5}$$

the pre-antisymplectic one-form potential (A.2) reduces to

$$\vartheta \;\sim\; \phi^*_\alpha d\phi^\alpha + \vartheta_a d\Theta^a \; . \tag{A.6}$$

We would like to show that the second term $\vartheta_a d\Theta^a$ in eq. (A.6) vanishes under a suitable anticanonical transformation. Eqs. (A.1) and (A.6) imply that the matrices $M_{a\alpha}$ and $N^\alpha{}_a$ in eq. (2.10) are

$$-M_{a\alpha} \;=\; (\vartheta_a \frac{\overleftarrow{\partial^r}}{\partial\phi^\alpha}) \;=\; (\vartheta_a, \phi^*_\alpha) \; , \tag{A.7}$$

$$N^\alpha{}_a \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\phi^*_\alpha}\vartheta_a) \;=\; (\phi^\alpha, \vartheta_a) \; , \tag{A.8}$$

and that the pre-antisymplectic potential components $\vartheta_a = \vartheta_a(\Gamma)$ satisfy a flatness condition:

$$F_{ab} \;\equiv\; (\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\vartheta_b) - (-1)^{\varepsilon_a\varepsilon_b}(\frac{\overrightarrow{\partial^l}}{\partial\Theta^b}\vartheta_a) + (\vartheta_a,\vartheta_b) \;=\; 0 \;. \tag{A.9}$$

Put more illuminating, the condition (A.9) implies that the vector fields

$$D_a \;\equiv\; \frac{\overrightarrow{\partial^l}}{\partial\Theta^a} + \mathrm{ad}\vartheta_a \tag{A.10}$$

commute

$$[D_a, D_b] \;=\; \mathrm{ad}F_{ab} \;=\; 0 \;. \tag{A.11}$$

Here the adjoint action "ad" refers to the antibracket $(\mathrm{ad}F)G \equiv (F,G)$, where F and G are functions. In other words, $\mathrm{ad}F$ denotes the Hamiltonian vector field with Hamiltonian F. The vector fields D_a are not Hamiltonian, although they do preserve the antibracket

$$D_a(F,G) \;=\; (D_a[F],G) + (-1)^{\varepsilon_a(\varepsilon_F+1)}(F,D_a[G]) \;, \tag{A.12}$$

i.e. they are generators of anticanonical transformations that do not leave the Casimirs invariant. It is an important fact that the D_a are covariant derivatives in the Casimir directions with a Lie algebra valued gauge potential $\mathrm{ad}\vartheta_a$. Here the Lie algebra is (a subalgebra of) the space $\Gamma(TM)$ of vector fields, equipped with the commutator $[\cdot,\cdot]$, *i.e.* the Lie bracket of vector fields. An infinitesimal variation $\delta\vartheta_a$ of the pre-antisymplectic potential components ϑ_a must satisfy

$$D_a[\delta\vartheta_b] \;=\; (-1)^{\varepsilon_a\varepsilon_b}(a \leftrightarrow b) \tag{A.13}$$

in order to respect the flatness condition (A.9). The last eq. (A.13) implies in turn, that the only allowed infinitesimal variations $\delta\vartheta_a$ are infinitesimal gauge transformations

$$\delta\vartheta_a \;=\; D_a[\delta\Psi] \;, \tag{A.14}$$

where $\delta\Psi$ is an infinitesimal fermionic gauge generator. The infinitesimal gauge transformation of the gauge potential $\mathrm{ad}\vartheta_a$ is

$$\mathrm{ad}(\delta\vartheta_a) \;=\; [D_a, \mathrm{ad}(\delta\Psi)] \;, \tag{A.15}$$

where use is made of eq. (A.12). Despite the appearance, the eq. (A.15) is exactly the standard formula $\delta A_\mu = D_\mu\varepsilon$ for infinitesimal non-Abelian gauge transformations. Any discrepancy is merely in notation, not in content. So one can take advantage of well-known facts about non-Abelian gauge theory and *e.g.* Wilson-lines. In particular, the infinitesimal transformations (A.14) and (A.15) generalize to finite gauge transformations. The field strength (or curvature) is zero, cf. eq. (A.11), so the gauge potential $\mathrm{ad}\vartheta_a$ is pure gauge. This means that there locally exists a gauge where the gauge potential vanishes identically,

$$\mathrm{ad}\vartheta_a \;=\; 0 \;. \tag{A.16}$$

An infinitesimal gauge transformation (A.14) may be implemented with the help of a Hamiltonian vector field $\mathrm{ad}(\delta\Psi)$ with infinitesimal Hamiltonian $\delta\Psi$. Using the active picture, the Lie derivative of the pre-antisymplectic one-form potential with respect to the Hamiltonian vector field $\mathrm{ad}(\delta\Psi)$ is

$$\mathcal{L}_{\mathrm{ad}(\delta\Psi)}\vartheta \;=\; [i_{\mathrm{ad}(\delta\Psi)}, d]\vartheta \;\sim\; i_{\mathrm{ad}(\delta\Psi)}E \;=\; (\delta\Psi\frac{\overleftarrow{\partial^r}}{\partial\gamma^A})d\gamma^a + (\delta\Psi,\vartheta_a)d\Theta^a \;\sim\; -D_a[\delta\Psi]d\Theta^a \;. \tag{A.17}$$

i.e. by flowing along the Hamiltonian vector field $\mathrm{ad}(\delta\Psi)$, one may mimic (minus) the infinitesimal gauge transformation (A.14). More generally, finite gauge transformations of ϑ_a are in one-to-one

correspondence with anticanonical transformations that leave the Casimirs invariant. In particular, one may go to the trivial gauge (A.16) where the ϑ_a themselves are Casimirs. The flatness condition (A.9) then reduces to

$$(\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\vartheta_b) \;=\; (-1)^{\varepsilon_a\varepsilon_b}(a\leftrightarrow b)\;, \tag{A.18}$$

so there exists a fermionic Casimir function $\Psi=\Psi(\Theta)$ such that

$$\vartheta_a \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\Psi)\;, \tag{A.19}$$

and hence the second term in eq. (A.6) is just an exact term,

$$\vartheta_a d\Theta^a \;=\; d\Psi \;\sim\; 0\;. \tag{A.20}$$

This shows that there locally exists an anticanonical transformation that leaves the Casimirs invariant, such that the two-form reduces to $E = d\phi^*_\alpha \wedge d\phi^\alpha$.

□

B Details from the Proof of Lemma 2.9

B.1 Proof of eq. (2.38)

The infinitesimal variation of $\nu^{(2)}$ in eq. (2.22) yields 8 contributions to linear order in the variation $\delta\Gamma^A = X^A$, which may be organized as 2×4 terms

$$\delta\nu^{(2)} \;=\; 2(-\delta\nu^{(2)}_I - \delta\nu^{(2)}_{II} + \delta\nu^{(2)}_{III} + \delta\nu^{(2)}_{IV})\;, \tag{B.1}$$

due to a $(A,B)\leftrightarrow(D,C)$ symmetry in eq. (2.22). They are

$$\begin{aligned}
\delta\nu^{(2)}_I &\equiv (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BF}(X^F\overleftarrow{\partial^r_C})(\overrightarrow{\partial^l_A}E^{CD})\;, && \text{(B.2)}\\
\delta\nu^{(2)}_{II} &\equiv (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}X^F)(\overrightarrow{\partial^l_F}E^{CD})\;, && \text{(B.3)}\\
\delta\nu^{(2)}_{III} &\equiv (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}\overrightarrow{\partial^l_A}\left((X^C\overleftarrow{\partial^r_F})E^{FD}\right) \;=\; \delta\nu^{(2)}_I + \delta\nu^{(2)}_V\;, && \text{(B.4)}\\
\delta\nu^{(2)}_{IV} &\equiv (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}\overrightarrow{\partial^l_A}\left(E^{CF}(\overrightarrow{\partial^l_F}X^D)\right) \;=\; \delta\nu^{(2)}_{II} + \delta\nu^{(2)}_{VI}\;, && \text{(B.5)}\\
\delta\nu^{(2)}_V &\equiv (-1)^{\varepsilon_A\varepsilon_C}E^{FD}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}X^C\overleftarrow{\partial^r_F}) \;=\; -\,\delta\nu^{(2)}_V + \delta\nu^{(2)}_{VII}\;, && \text{(B.6)}\\
\delta\nu^{(2)}_{VI} &\equiv (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_D}E^{AB})P_B{}^C(\overrightarrow{\partial^l_C}\overrightarrow{\partial^l_A}X^D)\;, && \text{(B.7)}\\
\delta\nu^{(2)}_{VII} &\equiv (-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C)\;, && \text{(B.8)}
\end{aligned}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the second equality of eq. (B.6). Altogether, the infinitesimal variation of $\nu^{(2)}$ becomes

$$\delta\nu^{(2)} \;=\; 2\delta\nu^{(2)}_{VI} + \delta\nu^{(2)}_{VII}\;, \tag{B.9}$$

which is eq. (2.38).

B.2 Proof of eq. (2.39)

The infinitesimal variation of $\nu^{(3)}$ in eq. (2.23) yields 6 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(3)} \;=\; \delta\nu^{(3)}_I + \delta\nu^{(3)}_{II} + \delta\nu^{(3)}_{III} - \delta\nu^{(3)}_{IV} - \delta\nu^{(3)}_{V} - \delta\nu^{(3)}_{VI} \;. \tag{B.10}$$

They are

$$\begin{aligned}
\delta\nu^{(3)}_I &\equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})(X^C\overleftarrow{\partial^r_F})E^{FD}(\overrightarrow{\partial^l_D}E^{BA}) \;, && \text{(B.11)}\\
\delta\nu^{(3)}_{II} &\equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\overrightarrow{\partial^l_D}\left((X^B\overleftarrow{\partial^r_F})E^{FA}\right) \;=\; \delta\nu^{(3)}_{VII} + \delta\nu^{(3)}_{VIII} \;, && \text{(B.12)}\\
\delta\nu^{(3)}_{III} &\equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\overrightarrow{\partial^l_D}\left(E^{BF}(\overrightarrow{\partial^l_F}X^A)\right) \;=\; \delta\nu^{(3)}_{IV} + \delta\nu^{(3)}_{IX} \;, && \text{(B.13)}\\
\delta\nu^{(3)}_{IV} &\equiv (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})(\overrightarrow{\partial^l_A}X^F)(\overrightarrow{\partial^l_F}E_{BC}) \;, && \text{(B.14)}\\
\delta\nu^{(3)}_{V} &\equiv (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})\overrightarrow{\partial^l_A}\left((\overrightarrow{\partial^l_B}X^F)E_{FC}\right) \;=\; \delta\nu^{(3)}_{VII} + \delta\nu^{(3)}_{X} \;, && \text{(B.15)}\\
\delta\nu^{(3)}_{VI} &\equiv (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})\overrightarrow{\partial^l_A}\left(E_{BF}(X^F\overleftarrow{\partial^r_C})\right) \;=\; \delta\nu^{(3)}_{I} - \delta\nu^{(3)}_{XI} \;, && \text{(B.16)}\\
\delta\nu^{(3)}_{VII} &\equiv (-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{AF})(\overrightarrow{\partial^l_F}X^B) \\
&= -(-1)^{\varepsilon_B(\varepsilon_C+1)}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})(\overrightarrow{\partial^l_A}E_{CF})(X^F\overleftarrow{\partial^r_B}) \;, && \text{(B.17)}\\
\delta\nu^{(3)}_{VIII} &\equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^B\overleftarrow{\partial^r_F})E^{FA} \;, && \text{(B.18)}\\
\delta\nu^{(3)}_{IX} &\equiv (-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^A\overleftarrow{\partial^r_F})E^{FB} \;, && \text{(B.19)}\\
\delta\nu^{(3)}_{X} &\equiv (-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C) \;, && \text{(B.20)}\\
\delta\nu^{(3)}_{XI} &\equiv (-1)^{\varepsilon_B(\varepsilon_C+1)}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_C}X^F)E_{FB} \;=\; -\,\delta\nu^{(3)}_{X} - \delta\nu^{(3)}_{XI} \;, && \text{(B.21)}
\end{aligned}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the second equality of eq. (B.21). Altogether, the infinitesimal variation of $\nu^{(3)}$ becomes

$$\delta\nu^{(3)} \;=\; \delta\nu^{(3)}_{VIII} + \delta\nu^{(3)}_{IX} - \frac{3}{2}\delta\nu^{(3)}_{X} \;, \tag{B.22}$$

which is eq. (2.39).

B.3 Proof of eq. (2.40)

The infinitesimal variation of $\nu^{(4)}$ in eq. (2.24) yields 6 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(4)} \;=\; -\,\delta\nu^{(4)}_I - \delta\nu^{(4)}_{II} + \delta\nu^{(4)}_{III} + \delta\nu^{(4)}_{IV} + \delta\nu^{(4)}_{V} - \delta\nu^{(4)}_{VI} \;. \tag{B.23}$$

They are

$$\begin{aligned}
\delta\nu^{(4)}_I &\equiv (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A\overrightarrow{\partial^l_A}\left((\overrightarrow{\partial^l_B}X^G)E_{GC}\right) \;=\; -\,\delta\nu^{(4)}_{VII} + \delta\nu^{(4)}_{VIII} \;, && \text{(B.24)}\\
\delta\nu^{(4)}_{II} &\equiv (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A\overrightarrow{\partial^l_A}\left(E_{BG}(X^G\overleftarrow{\partial^r_C})\right) \;=\; \delta\nu^{(4)}_{III} - \delta\nu^{(4)}_{IX} \;, && \text{(B.25)}
\end{aligned}$$

$$\delta\nu^{(4)}_{III} \equiv (-1)^{\varepsilon_B} P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})(X^C\overleftarrow{\partial^r_G})E^{GD}(\overrightarrow{\partial^l_D}E^{BF}) \ , \tag{B.26}$$

$$\delta\nu^{(4)}_{IV} \equiv (-1)^{\varepsilon_B} P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\overrightarrow{\partial^l_D}\left((X^B\overleftarrow{\partial^r_G})E^{GF}\right) = \delta\nu^{(4)}_{X} + \delta\nu^{(4)}_{XI} \ , \tag{B.27}$$

$$\delta\nu^{(4)}_{V} \equiv (-1)^{\varepsilon_B} P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\overrightarrow{\partial^l_D}\left(E^{BG}(\overrightarrow{\partial^l_G}X^F)\right) = \delta\nu^{(4)}_{VI} + \delta\nu^{(4)}_{XII} \ , \tag{B.28}$$

$$\delta\nu^{(4)}_{VI} \equiv (-1)^{\varepsilon_B} P_G{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{BF})(\overrightarrow{\partial^l_F}X^G) \ , \tag{B.29}$$

$$\delta\nu^{(4)}_{VII} \equiv (-1)^{\varepsilon_B(\varepsilon_C+1)} E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A(\overrightarrow{\partial^l_A}E_{CG})(X^G\overleftarrow{\partial^r_B}) \ , \tag{B.30}$$

$$\delta\nu^{(4)}_{VIII} \equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B}X^C)P_C{}^D(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A \ , \tag{B.31}$$

$$\delta\nu^{(4)}_{IX} \equiv (-1)^{\varepsilon_B(\varepsilon_C+1)} E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_C}X^G)E_{GB} = -\delta\nu^{(4)}_{VIII} - \delta\nu^{(4)}_{XIII} \ , \tag{B.32}$$

$$\delta\nu^{(4)}_{X} \equiv (-1)^{(\varepsilon_B+1)\varepsilon_F} P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{FG})(\overrightarrow{\partial^l_G}X^B) = -\delta\nu^{(4)}_{VII} \ , \tag{B.33}$$

$$\delta\nu^{(4)}_{XI} \equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^B\overleftarrow{\partial^r_F})E^{FA} \ , \tag{B.34}$$

$$\delta\nu^{(4)}_{XII} \equiv (-1)^{(\varepsilon_B+1)\varepsilon_F} P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^F\overleftarrow{\partial^r_G})E^{GB} \ , \tag{B.35}$$

$$\delta\nu^{(4)}_{XIII} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D}E^{BC})(\overrightarrow{\partial^l_C}\overrightarrow{\partial^l_A}X^G)E_{GB} = -\delta\nu^{(4)}_{XIII} - \delta\nu^{(4)}_{XIV} \ , \tag{B.36}$$

$$\delta\nu^{(4)}_{XIV} \equiv (-1)^{\varepsilon_A} P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C) \ , \tag{B.37}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the second equality of eqs. (B.32) and (B.36). Altogether, the infinitesimal variation of $\nu^{(4)}$ becomes

$$\delta\nu^{(4)} = -\delta\nu^{(4)}_{VIII} + \delta\nu^{(4)}_{IX} + \delta\nu^{(4)}_{XI} + \delta\nu^{(4)}_{XII} = -2\delta\nu^{(4)}_{VIII} + \delta\nu^{(4)}_{XI} + \delta\nu^{(4)}_{XII} + \frac{1}{2}\delta\nu^{(4)}_{XIV} \ , \tag{B.38}$$

which is eq. (2.39).

B.4 Proof of eq. (2.41)

The infinitesimal variation of $\nu^{(5)}$ in eq. (2.25) yields 8 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(5)} = \delta\nu^{(5)}_{I} + \delta\nu^{(5)}_{II} + \delta\nu^{(5)}_{III} - \delta\nu^{(5)}_{IV} - \delta\nu^{(5)}_{V} - \delta\nu^{(5)}_{VI} + \delta\nu^{(5)}_{VII} - \delta\nu^{(5)}_{VIII} \ . \tag{B.39}$$

They are

$$\delta\nu^{(5)}_{I} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B}(X^A\overleftarrow{\partial^r_G})E^{GD}(\overrightarrow{\partial^l_D}E^{BC})(\overrightarrow{\partial^l_C}E_{AF})P^F{}_B \ , \tag{B.40}$$

$$\delta\nu^{(5)}_{II} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B}(\overrightarrow{\partial^l_C}E_{AF})P^F{}_B E^{AD}\overrightarrow{\partial^l_D}\left((X^B\overleftarrow{\partial^r_G})E^{GC}\right) = \delta\nu^{(5)}_{VIII} + \delta\nu^{(5)}_{IX} \ , \tag{B.41}$$

$$\delta\nu^{(5)}_{III} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B}(\overrightarrow{\partial^l_C}E_{AF})P^F{}_B E^{AD}\overrightarrow{\partial^l_D}\left(E^{BG}(\overrightarrow{\partial^l_G}X^C)\right) = \delta\nu^{(5)}_{IV} - \delta\nu^{(5)}_{X} \ , \tag{B.42}$$

$$\delta\nu^{(5)}_{IV} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D}E^{BC})(\overrightarrow{\partial^l_C}X^G)(\overrightarrow{\partial^l_G}E_{AF})P^F{}_B \ , \tag{B.43}$$

$$\delta\nu^{(5)}_{V} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} P^F{}_B E^{AD}(\overrightarrow{\partial^l_D}E^{BC})\overrightarrow{\partial^l_C}\left((\overrightarrow{\partial^l_A}X^G)E_{GF}\right) = \delta\nu^{(5)}_{I} + \delta\nu^{(5)}_{XI} \ , \tag{B.44}$$

$$\delta\nu^{(5)}_{VI} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} P^F{}_B E^{AD}(\overrightarrow{\partial^l_D} E^{BC})\overrightarrow{\partial^l_C}\left(E_{AG}(X^G\overleftarrow{\partial^r_F})\right) = \delta\nu^{(5)}_{VII} - \delta\nu^{(5)}_{XII} , \quad \text{(B.45)}$$

$$\delta\nu^{(5)}_{VII} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} E^{BC})(\overrightarrow{\partial^l_C} E_{AF})(X^F\overleftarrow{\partial^r_G})P^G{}_B , \quad \text{(B.46)}$$

$$\delta\nu^{(5)}_{VIII} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} E^{BC})(\overrightarrow{\partial^l_C} E_{AF})P^F{}_G(X^G\overleftarrow{\partial^r_B}) , \quad \text{(B.47)}$$

$$\begin{aligned}\delta\nu^{(5)}_{IX} &\equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} X^B\overleftarrow{\partial^r_G})E^{GC}(\overrightarrow{\partial^l_C} E_{AF})P^F{}_B \\ &= (-1)^{\varepsilon_B\varepsilon_F} P_F{}^D(\overrightarrow{\partial^l_D} X^B\overleftarrow{\partial^r_G})E^{GC}(\overrightarrow{\partial^l_C} E^{FA})E_{AB} = \delta\nu^{(5)}_{XI} + \delta\nu^{(5)}_{XII} , \quad \text{(B.48)}\end{aligned}$$

$$\delta\nu^{(5)}_{X} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_C} (\overrightarrow{\partial^l_C} E_{AB})E^{BF}(\overrightarrow{\partial^l_F} X^C\overleftarrow{\partial^r_G})E^{GA} , \quad \text{(B.49)}$$

$$\delta\nu^{(5)}_{XI} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} E^{BC})(\overrightarrow{\partial^l_C}\overrightarrow{\partial^l_A} X^G)E_{GB} , \quad \text{(B.50)}$$

$$\delta\nu^{(5)}_{XII} \equiv (-1)^{\varepsilon_B} (\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B} X^C)P_C{}^D(\overrightarrow{\partial^l_D} E^{BF})P_F{}^A , \quad \text{(B.51)}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the third equality of eq. (B.48). Altogether, the infinitesimal variation of $\nu^{(5)}$ becomes

$$\delta\nu^{(5)} = \delta\nu^{(5)}_{IX} - \delta\nu^{(5)}_{X} - \delta\nu^{(5)}_{XI} + \delta\nu^{(5)}_{XII} = -\delta\nu^{(5)}_{X} + 2\delta\nu^{(5)}_{XII} , \quad \text{(B.52)}$$

which is eq. (2.41).

C Proof of Conversion Theorem 4.2

C.1 The $\bar{\jmath}$ Superdeterminant

Even-though it is only the $\jmath$-factor (4.13) and not the whole $\bar{\jmath}$ superdeterminant (4.14) that enters the conversion map, it is nevertheless convenient to organize the discussion in terms of coefficient functions for (the logarithm of) the $\bar{\jmath}$ superdeterminant

$$\ln\bar{\jmath} \equiv \bar{n} = n + \left\{\begin{array}{c} n^a_L\Phi_a \\ \Phi_a n^a_R \end{array}\right\} + \frac{1}{2}n^{ab}_L\Phi_b\Phi_a + \mathcal{O}(\Phi^3) , \qquad n \equiv \ln\jmath . \quad \text{(C.1)}$$

By combining eqs. (4.5), (4.14) and (C.1), one finds the first-order coefficient functions n^a to be

$$\begin{array}{rcccl} n^a_L &=& (-1)^{\varepsilon_b}X^R_{bc}Y^{cba}_M &=& (-1)^{\varepsilon_b+1}X^L_{bc}Y^{cba}_L \\ n^a_R &=& Y^{abc}_M X^L_{cb}(-1)^{\varepsilon_b} &=& Y^{abc}_R X^R_{cb}(-1)^{\varepsilon_b+1} . \end{array} \quad \text{(C.2)}$$

The second-order coefficient functions read

$$n^{cd}_L = (-1)^{\varepsilon_b+1}X^L_{ba}Z^{abcd}_L + (-1)^{(\varepsilon_a+1)\varepsilon_c}X^R_{ab}Y^{bce}_R X^R_{ef}Y^{fad}_M . \quad \text{(C.3)}$$

In particular, the contracted second-order coefficient function is

$$(-1)^{\varepsilon_c+1}n^{cd}_L\omega_{dc} = z^{(1)} - y^{(2)} , \quad \text{(C.4)}$$

where we have introduced the following short-hand notation

$$y^{(2)} \equiv (-1)^{\varepsilon_a\varepsilon_f}X^R_{ab}Y^{bfc}_M\omega_{cd}Y^{dae}_M X^L_{ef} , \quad \text{(C.5)}$$

$$z^{(1)} \equiv (-1)^{\varepsilon_b+\varepsilon_c}X^L_{ba}Z^{abcd}_L\omega_{dc} . \quad \text{(C.6)}$$

Since there is not a unique choice of the structure functions X^{ab}, Y^{abc}, Z^{abcd}, etc, one must apply the T^a involution relation (4.4) to eliminate their appearances. We have to wait until Subsection C.5 to

completely eliminate all Y^{abc} appearances, but we can do a first step in this direction. The quadratic Y^{abc} dependence inside the odd $y^{(2)}$ variable (C.5) can be related to a linear Y^{abc} dependence inside a new $y^{(1)}$ variable as follows

$$\begin{aligned}
0 &\stackrel{(4.4)}{=} \frac{1}{2}(-1)^{\varepsilon_a+1}Y_R^{bac}X_{cd}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_d}\, X_{ae}^L(T^e,T^f)_{\rm ext}X_{fb}^R\Big|_{\Phi=0} \\
&= \frac{1}{2}(-1)^{\varepsilon_c\varepsilon_e}X_{cd}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_d}\,(T^e,T^f)_{\rm ext}\Big|_{\Phi=0}X_{fb}^R Y_M^{bca}X_{ae}^L \\
&= (-1)^{\varepsilon_b}Y_R^{cba}(X_{ab}^R,\Theta^d)X_{dc}^R+(-1)^{\varepsilon_a\varepsilon_f}X_{ab}^R Y_M^{bfc}\omega_{cd}Y_M^{dae}X_{ef}^L \;=\; y^{(1)}+y^{(2)}\ ,
\end{aligned}\tag{C.7}$$

where

$$y^{(1)} \;\equiv\; (-1)^{\varepsilon_b}Y_R^{cba}(X_{ab}^R,\Theta^d)X_{dc}^R\ .\tag{C.8}$$

The only way the Z_L^{abcd} structure functions enters the discussion is through the odd $z^{(1)}$ variable (C.6). It can be eliminated using the following equation

$$\begin{aligned}
0 &\stackrel{(4.4)}{=} X_{dc}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_c}(-1)^{\varepsilon_a}X_{ab}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}\,(T^a,T^d)_{\rm ext}\Big|_{\Phi=0} \\
&= (-1)^{\varepsilon_a}X_{ab}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}\,(T^a,T^d)_{\rm ext}\frac{\overleftarrow{\partial^r}}{\partial\Phi_c}X_{cd}^L(-1)^{\varepsilon_d}\Bigg|_{\Phi=0} \\
&= (n,n)+n_L^a\omega_{ab}n_R^b+2(-1)^{\varepsilon_b+\varepsilon_c}X_{ba}^L(Y_L^{abc},\Theta^d)X_{dc}^R+(-1)^{\varepsilon_a\varepsilon_d}X_{ab}^R(X_R^{bd},X_L^{ac})X_{cd}^L \\
&\quad +2(-1)^{\varepsilon_b+\varepsilon_c}X_{ba}^L Z_L^{abcd}\omega_{dc}+(-1)^{\varepsilon_a\varepsilon_f}X_{ab}^R Y_M^{bfc}\omega_{cd}Y_M^{dae}X_{ef}^L \\
&= (n,n)+n_L^a\omega_{ab}n_R^b+2(n_R^a,\Theta^b)X_{ba}^R+(-1)^{\varepsilon_b}(X_L^{ab},X_{ba}^L)+2z^{(1)}+y^{(1)}\ .
\end{aligned}\tag{C.9}$$

C.2 Gauge Invariant Function $\bar{F}$

The gauge-invariant extension $\bar{F}$ is a power series expansion in the Φ_a variables, *e.g.* ,

$$\bar{F} \;=\; F+\left\{\begin{array}{c}\Phi_a F_R^a\\ F_L^a\Phi_a\end{array}\right\}+\frac{1}{2}\Phi_a\Phi_b F_R^{ba}+\mathcal{O}(\Phi^3)\ .\tag{C.10}$$

The coefficient functions for $\bar{F}$ are uniquely determined by gauge invariance condition (4.10). The first-order coefficient functions read

$$\begin{array}{rcccl}
F_R^a &=& -\omega^{ab}X_{bc}^L(\Theta^c,F) &=& X_R^{ab}E_{bc}(\Theta^c,F)\ ,\\
F_L^a &=& -(F,\Theta^c)X_{cb}^R\omega^{ba} &=& (F,\Theta^c)E_{cb}X_L^{ba}\ ,
\end{array}\tag{C.11}$$

The contracted second-order coefficient function $(-1)^{\varepsilon_a+1}\omega_{ab}F_R^{ba}$ is determined by the following calculation

$$\begin{aligned}
0 &\stackrel{(4.10)}{=} (-1)^{\varepsilon_a}X_{ab}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}\,(T^a,\bar{F})_{\rm ext}\big|_{\Phi=0} \\
&= X_{ba}^L(\Theta^a,F_R^b)(-1)^{\varepsilon_b+1}+(n,F)+n_L^a\omega_{ab}F_R^b+(-1)^{\varepsilon_a+1}\omega_{ab}F_R^{ba}\ .
\end{aligned}\tag{C.12}$$

C.3 Gauge Invariant Density $\bar{\rho}$

The (logarithm of the) gauge-invariant density $\bar{\rho}$ is a power series expansion in the Φ_a variables, *e.g.* ,

$$\ln\sqrt{\bar{\rho}} \;\equiv\; \bar{\ell} \;=\; \ell+\left\{\begin{array}{c}\ell_L^a\Phi_a\\ \Phi_a\ell_R^a\end{array}\right\}+\frac{1}{2}\Phi_a\Phi_b\ell_R^{ba}+\mathcal{O}(\Phi^3)\ , \qquad \ell \;\equiv\; \ln\sqrt{\rho j}\ .\tag{C.13}$$

The coefficient functions for $\bar{\rho}$ are uniquely determined by the gauge invariance condition (4.11). The first-order coefficient functions ℓ^a can be found from the following Lemma C.1.

Lemma C.1

$$
\begin{aligned}
\frac{1}{2}n^c_L - \ell^c_L &= (\Delta_\rho \Theta^a) X^R_{ab}\omega^{bc} + \frac{1}{2}(-1)^{\varepsilon_a}(\Theta^a, X^R_{ab})\omega^{bc} \\
&= \frac{(-1)^{\varepsilon_A}}{2\rho}\overrightarrow{\partial^l_A}\rho(\Gamma^A, \Theta^a) X^R_{ab}\omega^{bc} \;, \qquad \text{(C.14)} \\
\frac{1}{2}n^c_R - \ell^c_R &= \omega^{cb} X^L_{ba}(\Delta_\rho \Theta^a)(-1)^{\varepsilon_a} + \frac{1}{2}\omega^{cb}(X^L_{ba}, \Theta^a)(-1)^{\varepsilon_a} \\
&= \omega^{cb} X^L_{ba}(\Theta^a, \Gamma^A)\rho\overleftarrow{\partial^r_A}\frac{(-1)^{\varepsilon_A}}{2\rho} \;. \qquad \text{(C.15)}
\end{aligned}
$$

Proof of Lemma C.1: Combine

$$
0 \stackrel{(4.11)}{=} (\Delta_{\bar{\rho}} T^a)\Big|_{\Phi=0} = (\Delta_{\rho j}\Theta^a) + \frac{1}{2}(-1)^{\varepsilon_b+1}\omega_{bc}Y^{cba}_R + \ell^c_L\omega_{cb}X^{ba}_R \qquad \text{(C.16)}
$$

and

$$
\begin{aligned}
0 &\stackrel{(4.4)}{=} (-1)^{\varepsilon_b} X^R_{bc}\frac{\overrightarrow{\partial^l}}{\partial\Phi_c}(T^b, T^a)_{\rm ext}\Big|_{\Phi=0} \\
&= (-1)^{\varepsilon_c+1}X^L_{cb}(\Theta^b, X^{ca}_R) + (n, \Theta^a) + n^c_L\omega_{cb}X^{ba}_R + (-1)^{\varepsilon_b+1}\omega_{bc}Y^{cba}_R \;. \qquad \text{(C.17)}
\end{aligned}
$$

□

The contracted second-order coefficient function $(-1)^{\varepsilon_a+1}\omega_{ab}\ell^{ba}_R$ is determined by the following calculation

$$
\begin{aligned}
0 &\stackrel{(4.11)}{=} X^R_{ab}\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}(\Delta_{\bar{\rho}} T^a)\Big|_{\Phi=0} \\
&= (\Delta_{\rho j}X^{ab}_L)X^L_{ba}(-1)^{\varepsilon_a} + \frac{1}{2}(-1)^{\varepsilon_b+\varepsilon_c}X^L_{ba}Z^{abcd}_L\omega_{dc} \\
&\quad + X^L_{ba}(\Theta^a, \ell^b_L) + n^a_L\omega_{ab}\ell^b_R + (-1)^{\varepsilon_a+1}\omega_{ab}\ell^{ba}_R \;. \qquad \text{(C.18)}
\end{aligned}
$$

C.4 Assembling the Proof

Proof of eq. (4.25):

$$
(\bar{F}, \bar{G})_{\rm ext}\big|_{\Phi=0} = (F, G) + F^a_L\omega_{ab}G^b_R \stackrel{(C.11)}{=} (F, G)_D \;. \qquad \text{(C.19)}
$$

□

Proof of eq. (4.26):

$$
\begin{aligned}
(\Delta_{\bar{\rho}, E_{\rm ext}}\bar{F})\Big|_{\Phi=0} &= (\Delta_{\rho j}F) + \frac{1}{2}(-1)^{\varepsilon_a+1}\omega_{ab}F^{ba}_R + \ell^a_L\omega_{ab}F^b_R \\
&\stackrel{(C.12)}{=} (\Delta_{\rho j}F) - \frac{1}{2}(n, F) + (\ell^a_L - \frac{1}{2}n^a_L)\omega_{ab}F^b_R - \frac{1}{2}X^L_{ba}(\Theta^a, F^b_R)(-1)^{\varepsilon_b+1} \\
&\stackrel{(C.14)}{=} (\Delta_\rho F) - (\Delta_\rho\Theta^a)X^R_{ab}F^b_R - \frac{1}{2}(-1)^{\varepsilon_a}(\Theta^a, X^R_{ab}F^b_R) \\
&= (\Delta_\rho F) - (\Delta_\rho\Theta^a)E_{ab}(\Theta^b, F) - \frac{1}{2}(-1)^{\varepsilon_a}(\Theta^a, E_{ab}(\Theta^b, F)) \\
&= (\Delta_{\rho, E_D}F) \;. \qquad \text{(C.20)}
\end{aligned}
$$

$\square$

Proof of eq. (4.28): Using eq. (2.45) it follows that

$$\begin{aligned}
\nu_{\rho j} - \nu_{\rho} &\overset{(2.45)}{=} \frac{1}{\sqrt{j}}(\Delta_{\rho}\sqrt{j}) \;=\; \frac{1}{2}(\Delta_{\rho}n) + \frac{1}{8}(n,n) \;=\; \frac{1}{2}(\Delta_{\rho j}n) - \frac{1}{8}(n,n) \\
&= \frac{1}{2}(\Delta_{\rho j}X_{L}^{ab})X_{ba}^{L}(-1)^{\varepsilon_a} - \frac{1}{4}(-1)^{\varepsilon_b}(X_{L}^{ab},X_{ba}^{L}) - \frac{1}{8}(n,n)\ , \qquad \text{(C.21)}
\end{aligned}$$

so that

$$\begin{aligned}
\nu_{\bar{\rho},E_{\rm ext}}\Big|_{\Phi=0} &= \nu_{\rho j} + \frac{1}{2}(-1)^{\varepsilon_a+1}\omega_{ab}\ell_{R}^{ba} + \frac{1}{2}\ell_{L}^{a}\omega_{ab}\ell_{R}^{b} \\
&\overset{(C.21)}{=} \nu_{\rho} - \frac{1}{8}(n,n) + \frac{1}{2}(\Delta_{\rho j}X_{L}^{ab})X_{ba}^{L}(-1)^{\varepsilon_a} - \frac{1}{4}(-1)^{\varepsilon_b}(X_{L}^{ab},X_{ba}^{L}) \\
&\quad + \frac{1}{2}(-1)^{\varepsilon_a+1}\omega_{ab}\ell_{R}^{ba} + \frac{1}{2}\ell_{L}^{a}\omega_{ab}\ell_{R}^{b} \\
&\overset{(C.18)}{=} \nu_{\rho} - \frac{1}{8}(n,n) - \frac{1}{2}n_{L}^{a}\omega_{ab}\ell_{R}^{b} + \frac{1}{2}\ell_{L}^{a}\omega_{ab}\ell_{R}^{b} \\
&\quad - \frac{1}{4}(-1)^{\varepsilon_b}(X_{L}^{ab},X_{ba}^{L}) - \frac{z^{(1)}}{4} - \frac{1}{2}X_{ba}^{L}(\Theta^{a},\ell_{L}^{b}) \\
&\overset{(C.9)}{=} \nu_{\rho} + \frac{1}{2}(\frac{1}{2}n_{L}^{a} - \ell_{L}^{a})\omega_{ab}(\frac{1}{2}n_{R}^{b} - \ell_{R}^{b}) + \frac{1}{2}(\Theta^{a},\frac{1}{2}n_{L}^{b} - \ell_{L}^{b})X_{ba}^{L} \\
&\quad - \frac{1}{8}(-1)^{\varepsilon_b}(X_{L}^{ab},X_{ba}^{L}) + \frac{y^{(1)}}{8} \\
&\overset{(C.14)}{=} \nu_{\rho} - \frac{\nu_{\rho,D}^{(6)}}{2} + \frac{1}{2}(\Theta^{a},X_{ab}^{R})\omega^{bc}(\Delta_{\rho}\Theta^{c})(-1)^{\varepsilon_c} + \frac{1}{8}(-1)^{\varepsilon_a+\varepsilon_d}(\Theta^{a},X_{ab}^{R})\omega^{bc}(X_{cd}^{L},\Theta^{d}) \\
&\quad + \frac{1}{2}(\Theta^{a},\frac{1}{2}n_{L}^{b} - \ell_{L}^{b})X_{ba}^{L} - \frac{1}{8}(-1)^{\varepsilon_b}(X_{L}^{ab},X_{ba}^{L}) + \frac{y^{(1)}}{8} \\
&= \nu_{\rho} - \frac{\nu_{\rho,D}^{(6)}}{2} - \frac{\nu_{\rho,D}^{(7)}}{2} + \frac{1}{8}(-1)^{\varepsilon_a+\varepsilon_d}(\Theta^{a},X_{ab}^{R})\omega^{bc}(X_{cd}^{L},\Theta^{d}) \\
&\quad + \frac{1}{4}(-1)^{\varepsilon_b}(\Theta^{a},(\Theta^{b},X_{bc}^{R}))\omega^{cd}X_{da}^{L} - \frac{1}{8}(-1)^{\varepsilon_b}(X_{L}^{ab},X_{ba}^{L}) + \frac{y^{(1)}}{8} \\
&= \nu_{\rho,E_D} - \frac{\nu_{D}^{(9)}}{24} - \frac{x^{(1)}}{8} + \frac{y^{(1)}}{8} \overset{(C.23)}{=} \nu_{\rho,E_D}\ , \qquad \text{(C.22)}
\end{aligned}$$

where the last equality in eq. (C.22) follows from Lemma C.2 below, and the odd quantity $x^{(1)}$ is defined in eq. (C.25).

$\square$

C.5 Lemma C.2

It turns out that the most difficult part in the proof of eq. (4.28) is to eliminate the Y^{abc} dependence from the odd $y^{(1)}$ quantity (C.8). Lemma C.2 gives a formula for $y^{(1)}$ that are manifestly independent of Y^{abc}.

Lemma C.2

$$y^{(1)} \;=\; \frac{\nu_{D}^{(9)}}{3} + x^{(1)}\ . \qquad \text{(C.23)}$$

Proof of Lemma C.2: We first decompose the odd $\nu_D^{(9)}$ quantity (3.14) as

$$\nu_D^{(9)} \;\equiv\; (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}(\Theta^d, E_{ab})E^{bc}(E_{cd}, \Theta^a) \;=\; -\,x^{(1)} - 2x^{(2)} - x^{(3)} \;, \tag{C.24}$$

where

$$\begin{aligned}
x^{(1)} &\equiv (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}(\Theta^d, X^R_{ab})\omega^{bc}(X^L_{cd}, \Theta^a) \;, &\text{(C.25)}\\
x^{(2)} &\equiv (-1)^{\varepsilon_b(\varepsilon_d+1)}(\Theta^d, X^{bc}_L)(X^L_{cd}, \Theta^a)E_{ab} \;=\; (-1)^{\varepsilon_b(\varepsilon_d+1)}E_{ba}(\Theta^a, X^R_{dc})(X^{cb}_R, \Theta^d) \;, &\text{(C.26)}\\
x^{(3)} &\equiv (-1)^{\varepsilon_b\varepsilon_e}E_{ef}(\Theta^f, X^{bc}_L)\omega_{cd}(X^{de}_R, \Theta^a)E_{ab} \;. &\text{(C.27)}
\end{aligned}$$

Secondly, we define

$$\begin{aligned}
x^{(4)} &\equiv (-1)^{\varepsilon_c}E_{ab}(\Theta^b, X^{cd}_L)(X^L_{dc}, \Theta^a)\\
&= -(-1)^{\varepsilon_c}E_{ab}(\Theta^b, X^{cd}_R)(X^R_{dc}, \Theta^a) \;=\; x^{(3)} - x^{(1)} \;.
\end{aligned}\tag{C.28}$$

The third (=last) equality in eq. (C.28) is a non-trivial assertion. To prove it, we define the following quantities:

$$\begin{aligned}
x^{(5)} &\equiv (-1)^{\varepsilon_d}(\Theta^f, X^R_{dc})X^{cb}_R E_{ba}(\Theta^a, E^{de})E_{ef}\\
&= -(-1)^{\varepsilon_b\varepsilon_e}E_{ef}(\Theta^f, X^{bc}_L)X^L_{cd}(E^{de}, \Theta^a)E_{ab} \;=\; x^{(2)} + x^{(3)} \;, &\text{(C.29)}\\
x^{(6)} &\equiv (-1)^{\varepsilon_b\varepsilon_e}E_{ef}(\Theta^f, X^{bc}_L)X^L_{cd}(\Theta^d, E^{ea})E_{ab} \;=\; -\,x^{(1)} - x^{(2)} \;, &\text{(C.30)}\\
x^{(7)} &\equiv (-1)^{\varepsilon_d}(\Theta^f, X^R_{dc})X^{cb}_R E_{ba}(E^{ad}, \Theta^e)E_{ef} \;=\; -\,x^{(4)} + x^{(8)} \;, &\text{(C.31)}\\
x^{(8)} &\equiv (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}E^{ab}(X^R_{bd}, \Theta^e)E_{ef}(\Theta^f, X^R_{ac})\omega^{cd} \;=\; 0 \;, &\text{(C.32)}
\end{aligned}$$

where eq. (4.8) is used in the second equality of eqs. (C.29), (C.30) and (C.31). Remarkably the quantity $x^{(8)}$ vanishes due to an antisymmetry under the index permutation $ace \leftrightarrow bdf$. One may now check that the Jacobi identity

$$\sum_{\text{cycl. } a,b,c} (-1)^{(\varepsilon_a+1)(\varepsilon_c+1)}(E^{ab}, \Theta^c) \;=\; 0 \tag{C.33}$$

yields eq. (C.28):

$$0 \;=\; x^{(5)} + x^{(6)} + x^{(7)} \;=\; -\,x^{(1)} + x^{(3)} - x^{(4)} \;. \tag{C.34}$$

Thirdly, we define

$$\begin{aligned}
y^{(3)} &\equiv (-1)^{\varepsilon_c}E_{ab}Y^{bcd}_L\omega_{de}X^{ef}_R(X^R_{fc}, \Theta^a) \;=\; y^{(1)} + x^{(4)} + x^{(2)} \;, &\text{(C.35)}\\
y^{(4)} &\equiv (-1)^{(\varepsilon_a+1)\varepsilon_d}\omega^{ab}X^L_{bc}Y^{cde}_L\omega_{ef}X^{fg}_R(X^R_{ga}, \Theta^h)X^R_{hd} \;=\; y^{(1)} - x^{(1)} + x^{(2)} \;, &\text{(C.36)}
\end{aligned}$$

where eq. (4.9) is used in the second equality of eqs. (C.35) and (C.36). Note that $x^{(1)}$ to $x^{(8)}$ are manifestly independent of the Y^{abc} structure functions. We shall soon see that this is also the case for the variables $y^{(1)}$ to $y^{(4)}$. It turns out to be possible to rewrite $y^{(3)}$ as

$$\begin{aligned}
y^{(3)} &= \frac{1}{2}(-1)^{(\varepsilon_a+1)(\varepsilon_c+\varepsilon_f+1)+\varepsilon_c}E_{ab}Y^{bcd}_L X^L_{de}(E^{ef}, \Theta^a)X^R_{fc}\\
&= (-1)^{\varepsilon_a\varepsilon_c}E_{ab}Y^{bcd}_L X^L_{de}(E^{ea}, \Theta^f)X^R_{fc} \;=\; -\,y^{(1)} - y^{(4)} \;.
\end{aligned}\tag{C.37}$$

Here the Jacobi identity (C.33) is used in the second equality of eq. (C.37). Altogether, eqs. (C.35), (C.36) and (C.37) yields

$$3y^{(1)} \;=\; x^{(1)} - 2x^{(2)} - x^{(4)} \;. \tag{C.38}$$

Now Lemma C.2 follows by combining eqs. (C.24), (C.28) and (C.38).

□

References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **102B** (1981) 27.

[2] I.A. Batalin and G.A. Vilkovisky, Phys. Rev. **D28** (1983) 2567 [E: **D30** (1984) 508].

[3] I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. **B234** (1984) 106.

[4] K. Bering, arXiv:physics/9711010.

[5] A. Schwarz, Commun. Math. Phys. **155** (1993) 249.

[6] I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. **A8** (1993) 2333.

[7] I.A. Batalin, K. Bering and P.H. Damgaard, Phys. Lett. **B389** (1996) 673.

[8] I.A. Batalin, K. Bering and P.H. Damgaard, Nucl. Phys. **B739** (2006) 389.

[9] K. Bering, J. Math. Phys. **47** (2006) 123513, arXiv:hep-th/0604117.

[10] O.M. Khudaverdian, arXiv:math.DG/9909117.

[11] O.M. Khudaverdian and Th. Voronov, Lett. Math. Phys. **62** (2002) 127.

[12] O.M. Khudaverdian, Contemp. Math. **315** (2002) 199.

[13] O.M. Khudaverdian, Commun. Math. Phys. **247** (2004) 353.

[14] I.A. Batalin, K. Bering and P.H. Damgaard, Phys. Lett. **B408** (1997) 235.

[15] I.A. Batalin and E.S. Fradkin, Nucl. Phys. **B279** (1987) 514; Phys. Lett. **B180** (1986) 157; E: *ibid* **B236** (1990) 528.

[16] I.A. Batalin, E.S. Fradkin, and T.E. Fradkina, Nucl. Phys. **B314** (1989) 158; E: *ibid* **B323** (1989) 734; *ibid* **B332** (1990) 723.

[17] I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. **A6** (1991) 3599.

[18] E.S. Fradkin and V.Ya. Linetskii, Nucl. Phys. **B431** (1994) 569; *ibid* **B444** (1995) 577.

[19] I.A. Batalin and R. Marnelius, Nucl. Phys. **B511** (1998) 495.

[20] K. Bering, P.H. Damgaard and J. Alfaro, Nucl. Phys. **B478** (1996) 459.

[21] K. Bering, Commun. Math. Phys. **274** (2007) 297; arXiv:hep-th/0603116.

[22] I.A. Batalin and K. Bering, J. Math. Phys. **49** (2008) 033515, arXiv:0708.0400.